3.Complex Number.

Complex Numbers

Definitions.

$$\text{Let } i^2 = -1.$$
$$\therefore\ i = \sqrt{-1}.$$

Complex numbers are often denoted by z.

Just as $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. If z is a complex number, z is of the form

$$z = x + iy \in \mathbb{C},\ \text{for some } x, y \in \mathbb{R}.$$

e.g. $3 + 4i$ is a complex number.

$$\underset{\text{real part}}{x} \quad \underset{\text{imaginary part.}}{iy} \qquad z = x + iy$$

If $z = x + iy$, $x, y \in \mathbb{R}$,

$$\textit{the real part of } z = \Re(z) = \mathrm{Re}(z) = x$$
$$\textit{the imaginary part of } z = \Im(z) = \mathrm{Im}(z) = y.$$

eg. $z = 3 + 4i$

$\Re(z) = 3$

$\Im(z) = 4.$

Example// Express each of the following in terms of i.

1. $\sqrt{-25} \rightarrow \sqrt{25i^2} = 5i$

2. $\sqrt{-12} - \sqrt{-3} \rightarrow \sqrt{12i^2} - \sqrt{3i^2} = 2\sqrt{3}i - \sqrt{3}i = \sqrt{3}i$

3. $-2+\sqrt{-4} \rightarrow -2 + \sqrt{4i^2} = -2+2i$

0peration of complex number:-

ADDITION OF COMPLEX NUMBERS

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

SUBTRACTION OF COMPLEX NUMBERS

$$(a+bi)-(c+di) = (a-c)+(b-d)i$$

MULTIPLICATION OF COMPLEX NUMBERS

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

DIVISION OF COMPLEX NUMBERS

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2}+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

Example 1. $(2+3i)+(4+i)=6+4i$.
Example 2. $(8-3i)-(-2+4i)=10-7i$.

Multiplication/Division.

Example 1. $(2+3i)(1+2i)=2+4i+3i-6=-4+7i$
Example 2. $(3-2i)(3+2i)=9-(2i)^2=9+4=13$

$\therefore$ when we multiply two complex conjugates, we get a real number.

Example 3. $\frac{2+3i}{1+4i}=\frac{2+3i}{1+4i}\times\frac{1-4i}{1-4i}=\frac{(2+3i)(1-4i)}{(1+4i)(1-4i)}=\frac{2-8i+3i-12i^2}{1-(4i)^2}=\frac{14-5i}{17}$

ABSOLUTE VALUE

The *absolute value* or *modulus* of a complex number $a+bi$ is defined as $|a+bi| = \sqrt{a^2+b^2}$.

Example: $|-4+2i| = \sqrt{(-4)^2+(2)^2} = \sqrt{20} = 2\sqrt{5}$

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $|z_1+z_2| \leq |z_1|+|z_2|$
4. $|z_1+z_2| \geqq |z_1|-|z_2|$

Example//show that $|Z1.Z2| = |Z1|.|Z2|$

Solution//

Let $z_1=a_1+b_1i$, $z_2=a_2+b_2i$

$|(a1+b1i).(a2+b2i)| = |(a1a2-b1b2)+(a1b2+b1a2)i|$

$=\sqrt{(a1a2-b1b2)^2+(b2a1+b1a2)^2}$

$\sqrt{(a1^2a2^2-2a1a2b1b2+b1^2b2^2+a1^2b2^2+2a1a2b1b2+b1^2a2^2)}$

$\sqrt{(a1^2a2^2+b1^2b2^2+a1^2b2^2+b1^2a2^2)}$

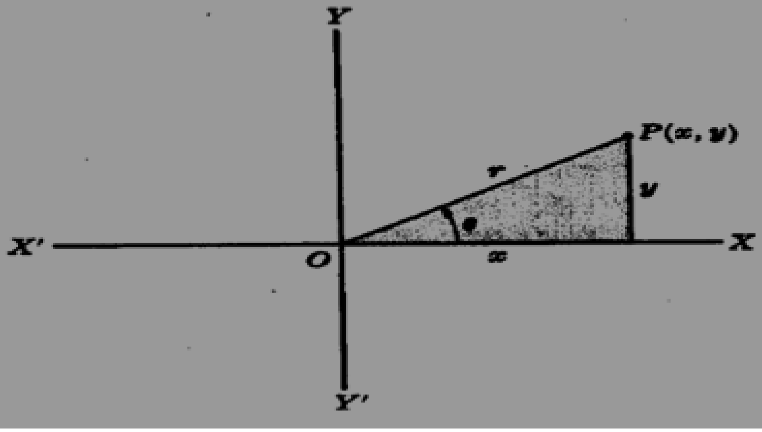
While $|Z1|.|Z2| = |a1+b1i|.|a2+b2i|$

$=\sqrt{a1^2+b1^2}.\sqrt{a2^2+b2^2}$

$=\sqrt{(a1^2a2^2+b1^2b2^2+a1^2b2^2+b1^2a2^2)}$

POLAR FORM OF COMPLEX NUMBERS

If P is a point in the complex plane corresponding to the complex number (x, y) or $x + iy$,



where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of $z = x + iy$ [denoted by *mod* z or $|z|$]; and θ, called the *amplitude* or *argument* of $z = x + iy$ [denoted by *arg* z], is the angle which line OP makes with the positive x axis.

It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta) \qquad (1)$$

which is called the *polar form* of the complex number, and r and θ are called *polar coordinates*. It is sometimes convenient to write the abbreviation $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$.

$X = r\cos\theta \;, y = r\sin\theta$

Example// Express the polar form of 3+3i.

Solution/ z=3+3i=r$[\cos\theta + i\sin\theta]$

X=$r\cos\theta = 3$,$y = r\sin\theta = 3$

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = \pm 3\sqrt{2}$

X=$r\cos\theta \rightarrow 3 = 3\sqrt{2}\cos\theta \rightarrow \theta = 45$

z=3+3i=$3\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

Eulers theorem :-

$$e^{i\theta} = (\cos\theta + i\sin\theta)] * r$$

$re^{i\theta} = r(\cos\theta + i\sin\theta)$........1

$re^{-i\theta} = r(\cos\theta + i\sin\theta)$......2

Adding 1 and 2

$$re^{i\theta} + re^{-i\theta} = 2r\cos \rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{sub in1}$$

$$\rightarrow \sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$$

Example// show that $\cos^2\theta + \sin^2\theta = 1$

Solution //

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$$

$$\cos^2\theta + \sin^2\theta = \frac{e^{2i\theta}}{4} + e^0 + \frac{e^{-2i\theta}}{4} + \frac{-e^{-2i\theta}}{4} + e^0 + \frac{-e^{-2i\theta}}{4}$$

$$e^0+e^0 = 1+1 = 2$$