9.Integration by partial fraction method:-

Using when partial fraction only formula $\frac{f(x)}{g(x)}$.

When the degree of the numerator is less than the degree of the denominator. There are four states.

First state:-

If the factors of g (x) are linear and not repeated.

$$\int \frac{dx}{(ax+b)(ax-b)} = \int \frac{A}{(ax+b)}dx + \int \frac{B}{(ax-b)}dx$$

Example 1: Find $\int \frac{5X-3}{(X+1)(X-3)}dx$ **.**

Solution /

$$\int \frac{5X-3}{(X+1)(X-3)}dx = \int \frac{A}{(X+1)}dx + \int \frac{B}{(X-3)}dx$$

$$\frac{5X-3}{(X+1)(X-3)} = \frac{A(X-3)+B(X+1)}{(X+1)(X-3)}$$

$$5X-3 = AX-3A+BX+B$$

A+B=5……….(1)

B-3A=-3………(2)

5-A-3A=-3$\rightarrow A = 2 \rightarrow B = 3$

$$\int \frac{5X-3}{(X+1)(X-3)}dx = \int \frac{2}{(X+1)}dx + \int \frac{3}{(X-3)}dx$$

$$=2\ \ln|x+1|+3ln|x-3|+c$$

H.W. //1. $\int \frac{5x-7}{(x-1)(x-2)}dx$

Second state: -

If the factors of g (x) are linear and repeated

$$\int \frac{dx}{(ax+b)^2}=\int \frac{A}{(ax+b)^1}dx+\int \frac{B}{(ax-b)^2}dx$$

Example 2: ***Find*** $\int \frac{x^2}{(x-1)(x+1)^2}\ dx.$

Solution /

$$=\int \frac{x^2}{(x-1)(x+1)^2}\ dx=\int \frac{A}{(X-1)}dx+\int \frac{B}{(X+1)}dx+\int \frac{c}{(X+1)^2}dx$$

$$\frac{x^2}{(x-1)(x+1)^2}=\frac{A(X+1)^2+B(X-1)(X+1)+C(X-1)}{(x-1)(x+1)^2}$$

$$X^2=AX^2+2AX+A+BX^2-B+CX-C$$

A+B=1…….(1)

2A+C=0...…(2)

A-B-C=0......(3)

From (2)&(3) A-B+2A=0 $\rightarrow 3A-B=0$ **SUB IN** $\ldots(1)$

A=$\frac{1}{4}$ $.B=\frac{3}{4}$ $.C=\frac{-1}{2}\therefore$

$$= \int \frac{\frac{1}{4}}{(X-1)} dx + \int \frac{\frac{3}{4}}{(X+1)} dx + \int \frac{\frac{-1}{2}}{(X+1)^2} dx$$

$$\frac{1}{4} ln|x-1| + \frac{3}{4} ln|x+1| + \frac{1}{2(x+1)} + c$$

Third state:-

If the factors of g (x) are not linear and not repeated.

$$\int \frac{dx}{(ax^2+b)(ax^2-b)} = \int \frac{Ax+B}{(ax^2+b)} dx + \int \frac{Cx+D}{(ax^2-b)} dx$$

Example 3: Find $\int \frac{1}{(x^2+1)(x+1)} dx$

Solution/

$$\int \frac{1}{(x^2+1)(x+1)} dx = \int \frac{Ax+B}{(x^2+1)} dx + \int \frac{C}{(x+1)} dx$$

$$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2 + C$$

A+C=0…….(1)

A+B=0...…(2)

B+C=1......(3)

From (1) ,(2)&(3)

A=$\frac{-1}{2}$ $.B=\frac{1}{2}$ $.C=\frac{1}{2}\therefore$

$$\int\frac{1}{(x^2+1)(x+1)}dx=\int\frac{\frac{-1}{2}x+\frac{1}{2}}{(x^2+1)}dx+\int\frac{\frac{1}{2}}{(x+1)}dx$$

$=\frac{-1}{2}\int\frac{x}{(x^2+1)}dx+\frac{1}{2}\int\frac{dx}{(x^2+1)}dx+\frac{1}{2}\int\frac{dx}{(x+1)}$

$=\frac{-1}{4}\ln|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}ln|x+1|+c$

Four state: -

If the factors of g (x) are not linear and repeated.

$$\int\frac{dx}{(ax^2+b)^2}=\int\frac{Ax+B}{(ax^2+b)^1}dx+\int\frac{Cx+D}{(ax^2+b)^2}dx$$

Example 4: Find$\int\frac{(x^2+1)}{(x^2+2x+3)^2}dx$

Solution/

$$\int\frac{(x^2+1)}{(x^2+2x+3)^2}dx$$
$$=\int\frac{Ax+B}{(x^2+2x+3)}dx+\int\frac{Cx+D}{(x^2+2x+3)^2}dx$$

$$\frac{(x^2+1)}{(x^2+2x+3)^2}=\frac{Ax+B(x^2+2x+3)+Cx+D}{(x^2+2x+3)^2}$$

$$x^2+1=Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx+D$$

A=0

2A+B=1→B=1

3A+2B+C=0→C=-2

3B+D=1→ $\boldsymbol{D=-2}$

$$\int \frac{(x^2+1)}{(x^2+2x+3)^2}dx$$
$$=\int \frac{1}{(x^2+2x+3)}dx+\int \frac{-2x-2}{(x^2+2x+3)^2}dx$$

$$=\int \frac{1}{(x+1)^2+(\sqrt{2})^2}dx+\int(-2x-2)(x^2+2x+3)^{-2}\,dx$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x+1)}{\sqrt{2}}+\frac{(x^2+2x+3)^{-1}}{1}+c$$