Al-Mustaqbal University College

Chemical Engineering and Petroleum Industries department

The Binomial Distribution

* If P is the prob. That an event will happen in any single trial (called the prob. Of succeed)and q=1−P is the prob. That it will fail to happen in any single trial (called the prob. Of a failure), then the prob. That the event will happen exactly X times in N trials (X success and N−X failures will occur) is given by :

$$P(X) = {}_{N}C_{X}\, P^{X}\, q^{N-X} = \frac{N\,!}{X\,!\,(N-X)!}\; P^{X}\, q^{N-X}$$

Where :

X = 0 , 1 , 2 , … , N

N ! = N (N−1) (N−2) …1

0 ! = 1

3! = 3*2*1 =6

10! = 10*9*8*7*6*5*4*4*3*2*1= 3,628,800

Example 1)

The prob. Dist. For getting heads in N tossed of a coin is :

$$P(X) = {}_NC_X \left(\frac{1}{2}\right)^X \left(\frac{1}{2}\right)^{N-X}$$

For example N=3 ?

Pr (0 heads) = ${}_3C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0}$

$$= \frac{3\,!}{0\,!\,(3-0)!} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0} = \frac{1}{8}$$

Pr (1 heads) = ${}_3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{3-1} = \frac{3}{8}$

Pr (2 heads) = ${}_3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} = {}_3C_2 \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

Pr (3 heads) = ${}_3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} = {}_3C_3 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

H = head , T = tail

HHH , HHT , HTT , TTT , THH , TTH , THT , HTH

Total out coins = 8 = $(2)^N = 2^3$ =8

Where 2 = P(head + tail)

Example 2)

Find the prob. Of getting :

a) Pr (2 H in 6 tosses) = ${}_6C_2 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{6-2}$

$$= \frac{6\,!}{2\,!\,(6-2)!} \left(\frac{1}{2}\right)^6 = \frac{15}{64}$$

Total out coins : $(2)^6 = 64$

b) Pr (at least 4 H in 6 tosses) = Pr (X≥4)

= Pr (X=4) + Pr (X=5) + Pr (X=6)

∴ $\Pr(X \geq 4) = [\,{}_6C_4 + {}_6C_5 + {}_6C_6]\,[\frac{1}{2}]^6 = \frac{22}{64}$

Example 3)

If 10% of items produced by a machine are defective, find the prob. that out of 5 items :

a) None are defective

P = prob. of success (defection) .

P = 0.1 → q + p =1 → q=0.9

Q = 0.9 = prob. of failure (non def.)

Pr (o def.) = Pr (X=0) = ${}_5C_0\,(0.1)^0\,(0.9)^5$

$$\Pr(X=0) = \frac{5\,!}{0\,!\,(5-0)!}\,(0.1)^0\,(0.9)^5 = 0.5905$$

b) All are def.

Pr (5 def.) = Pr (X=5) = ${}_5C_5\ (0.1)^5\ (0.9)^5$

$$= \frac{5\,!}{5\,!\,(5-5)!}\ (0.1)^5\ (0.9)^0 = 0.00001$$

c) At most 2 def.

Pr (at most 2 def.) = Pr (X≤2) = Pr (X=2) + Pr (X=1)+ Pr (X=0)

Pr (X≤2) = ${}_5C_2\ (0.1)^2\ (0.9)^3 +\ {}_5C_1\ (0.1)^1\ (0.9)^4 +$

$${}_5C_0\ (0.1)^0\ (0.9)^5$$

$$= 0.00729 + 0.32805 + 0.59049 = 0.9258$$

Some properties of Binomial dist :

Mean = μ = NP

Variance = σ^2 = NPq

Stand. dev. = $\sigma = \sqrt{N\,P\,q}$

Relation between Binomial and Normal Distributions

* If N is large and if neither P nor q is too close to zero, the binomial distribution can be closely approximated by a normal distribution with standardized variable given by :

$$Z = \frac{X - \mu}{\sigma} = \frac{X - NP}{\sqrt{N\,P\,q}}$$

The approximation becomes better with increasing N , . In practice, the approximate is good when : NP > 5 , Nq > 5 .

Example 1)

Find the prob. of obtaining 3 – 6 heads in 10 tosses of a coin :

a) using the Binomial dist.

Pr(3-6 heads) = Pr(3) + Pr(4) + Pr(5) + Pr(6)

$$= \left[{}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6\right]\left[\frac{1}{2}\right]^{10} = 0.7734$$

b) using the normal dist.

one unit = 1.0

Pr (3-6 heads) = Pr $(3 \le X \le 6)$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - NP}{\sqrt{N\,P\,q}}$$

μ = NP = 10*0.5 = 5

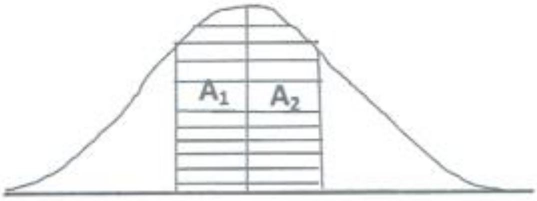
$\sigma = \sqrt{N\,P\,q} = \sqrt{10 * 0.5^2} = 1.581$

For $X_1 = 2.5 \rightarrow Z_1 = -1.581 \rightarrow A_1 = 0.4431$

$X_2 = 6.5 \rightarrow Z_2 = 0.949 \rightarrow A_2 = 0.3287$

Pr $(-1.581 < Z < 0.949) = A_1 + A_2 = 0.7718$

% of error between the binomial and normal = 0.0016



Example 2)

What is the prob. That at most 90% of 20 students will graduate ? Given % of graduate 70% ?

a) using the Binomial dist. :

Pr (at most $\frac{90}{100} * 20$) = Pr (at most 18)

Pr $(X \leq 18)$ = Pr(0) + Pr(1) + … + Pr(18)

$= 1-[Pr(19) + Pr(20)]$

Pr (X=19) = $_{20}C_{19}\ (0.7)^{19}\ (0.3)^{1} = 6.84*10^{-3}$

Pr (X=20) = $_{20}C_{20}\ (0.7)^{20}\ (0.3)^{0} = 7.98*10^{-4}$

Pr (X≤18) = $1-(6.84*10^{-3} + 7.98*10^{-4}) = 0.992$

b) using the normal dist.

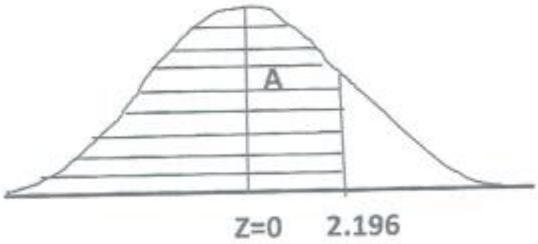
$$Z = \frac{X-\mu}{\sigma} = \frac{X-NP}{\sqrt{N\,P\,q}}$$

$\mu = NP = 20*0.7 = 14$

$\sigma = \sqrt{N\,P\,q} = 2.049$

Pr (X≤18) → X = 18.5 → Z = 2.196 → A = 0.48

Pr (Z < 2.196) = 0.5 + 0.486 = 0.986



Relation between Binomial and Poisson :

Poisson dist. :

Is a discrete prob. dist. Defined by :

$$Pr(X) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Where :

X = 0 , 1 , 2, … , N

e = 2.71828

With properties :

$\mu = \lambda$, $\sigma = \sqrt{\lambda}$

Example 1)

Product of a machine is 10% defective, Find the prob. of obtaining at most 2 def. items out of 10 :

a) using Binomial :

Pr [at most 2] = Pr($X \leq 2$)=Pr(0) + Pr(1) + Pr(2)

$= {}_{10}C_0\,(0.1)^0\,(0.9)^{10} + {}_{10}C_1\,(0.1)^1\,(0.9)^9 + {}_{10}C_2\,(0.1)^2\,(0.9)^8$

Pr ($X \leq 2$) = 0.9298

b) using Poisson :

Poisson is applicable for large N , While P is close to zero.

In practice , N≥50 , NP < 5 .

Use $\mu = NP = \lambda = 10 * 0.1 = 1$

Pr (at most 2) = Pr(X≤ 2) = Pr(0) + Pr (1) +Pr(2)

$$= \frac{1^0\, e^{-1}}{0\,!} + \frac{1^1\, e^{-1}}{1\,!} + \frac{1^2\, e^{-1}}{2\,!} = \left[1 + 1 + \frac{1}{2}\right] e^{-1}$$

$$= 0.9197$$

Example 2)

The prob. of failure of a certain process is 3% . Determine the prob. of 3 failures at most in 100 repetition of the process :

a) using binomial dist.

Pr (at most 3) = Pr($X \leq 3$)=

$${}_{100}C_0\,(0.03)^0\,(0.97)^{100} + \cdots + {}_{100}C_3\,(0.03)^3\,(0.97)^{97}$$

= 0.6474

b) using Poisson dist. :

N = 100 , NP = = $100 * \frac{3}{100}$ = 3 = λ

$$\Pr(X \leq 3) = \frac{3^0\, e^{-3}}{0\,!} + \cdots + \frac{3^3\, e^{-3}}{3\,!}$$

$$= \left[\frac{3^0}{0\,!} + \frac{3^1}{1\,!} + \frac{3^2}{2\,!} + \frac{3^3}{3\,!}\right] [e^{-3}] = 0.6472$$