

7. Limits and their properties, definition and theory.**-Properties of limits**

$$1. \lim_{x \rightarrow a} x = a$$

$$2. \lim_{x \rightarrow a} c = c$$

$$3. \lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$6. \lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

$$7. \lim_{x \rightarrow 0} \frac{1}{x} = \infty$$

$$8. \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$9. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$10. \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$11. \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

Example 1: Find $\lim_{x \rightarrow 0} x^2 + 3x + 5$

Solution/ /

$$= 0 + 3 * 0 + 5 = 5$$

Example 2: Find $\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$

Solution //

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)} &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x \rightarrow 1} \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)} \end{aligned}$$

$$\lim_{x \rightarrow 1} \frac{(2 * 1 - 3)}{(2 * 1 + 3)(\sqrt{1} + 1)} = -1/10$$

Example 3: Find $\lim_{x \rightarrow 2} \frac{(x^2-4)}{(x^3-8)}$

Solution //

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{(x^2-4)}{(x^3-8)} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} \\ &= \lim_{x \rightarrow 2} \frac{(x+2)}{(x^2+2x+4)} = \lim_{x \rightarrow 2} \frac{(2+2)}{(2^2+2*2+4)} \\ &= 1/3 \end{aligned}$$

Example 4: Find $\lim_{x \rightarrow 2} \frac{(4-x^2)}{(3-\sqrt{x^2+5})}$

Solution /

$$\lim_{x \rightarrow 2} \frac{(4-x^2)}{(3-\sqrt{x^2+5})} * \frac{(3+\sqrt{x^2+5})}{(3+\sqrt{x^2+5})} = \lim_{x \rightarrow 2} \frac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5}) * (3+\sqrt{x^2+5})}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 2} \frac{(4 - x^2)(3 + \sqrt{x^2 + 5})}{9 - (x^2 + 5)} = \\
 &= \lim_{x \rightarrow 2} \frac{(4 - x^2)(3 + \sqrt{x^2 + 5})}{(4 - x^2)} = (3 + \sqrt{x^2 + 5}) \\
 &= 3 + \sqrt{4 + 5} = 6
 \end{aligned}$$

Example 5: Find $\lim_{x \rightarrow \infty} \frac{(x+2)}{\sqrt{x^2+1}}$

Solution //

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x}{x^2} + \frac{2}{x^2}\right)}{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x} + \frac{2}{x^2}\right)}{\sqrt{1 + \frac{1}{x^2}}} = \frac{0}{1} = 0$$

Example 6: Find $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right)$

Solution //

$$\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right) = \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{(x-2)(x+2)}\right)$$

$$\begin{aligned}
 \lim_{x \rightarrow 2} \left(\frac{1}{x-2} \left(1 - \frac{4}{(x+2)}\right)\right) &= \lim_{x \rightarrow 2} \frac{(x+2-4)}{(x-2)(x+2)} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)}{(x-2)(x+2)} \\
 &= \frac{1}{2+2} = 1/4
 \end{aligned}$$

Example 7: Find $\lim_{h \rightarrow 0} \left(\frac{\sin(a+h) - \sin a}{h} \right)$

Solution //

$$\lim_{h \rightarrow 0} \left(\frac{\sin(a) \cos(h) + \cos(a) \sin(h) - \sin(a)}{h} \right)$$

$$\lim_{h \rightarrow 0} \left(\frac{\sin(a) ((\cos(h) - 1) + \cos(a) \sin(h))}{h} \right)$$

$$\lim_{h \rightarrow 0} \left(\frac{-\sin(a) (1 - \cos(h)) + \cos(a) \sin(h)}{h} \right)$$

$$\lim_{h \rightarrow 0} \left(\frac{-\sin(a) (1 - \cos(h))}{h} \right) + \lim_{h \rightarrow 0} \left(\frac{\sin(h) \cos(a)}{h} \right) = \cos a$$

Example 8: Find $\lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})}{x^2}$

Solution //

$$\lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})}{x^2} * \frac{(1 + \sqrt{\cos x})}{(1 + \sqrt{\cos x})}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 (1 + \sqrt{\cos x})} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 (1 + \sqrt{\cos x})} * \frac{1 + \cos x}{(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2 ((1 + \cos x)(1 + \sqrt{\cos x}))}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2 (1 + \cos x)(1 + \sqrt{\cos x})}$$

$$= 1/4$$

Example 9: Find $\lim_{x \rightarrow \pi} \left(\frac{\sin(x)}{x - \pi} \right)$

Solution /

$$Y = x - \pi \rightarrow x = y + \pi . y = 0$$

$$\lim_{y \rightarrow 0} \left(\frac{\sin(y + \pi)}{y} \right) = \lim_{y \rightarrow 0} \left(\frac{\sin(y) \cos(\pi) + \sin(\pi) \cos(y)}{y} \right)$$

$$= -1$$

H.W. Example 1: Find $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan X)\right)$

Example 2: Find $\lim_{x \rightarrow 1} \frac{(\sqrt{x} - 2)}{(x - 2)}$

Example 3: Find $\lim_{x \rightarrow 1} \frac{(x^2 - 9)}{(\sqrt{x} + 3)}$

Example 4: Find $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - X)$

Example 5: Find $\lim_{x \rightarrow 0} \frac{(X * \sin X)}{(2 - 2 \cos X)}$

Example 6: Find $\lim_{x \rightarrow \infty} \frac{(x^2 + 2X + 1)^{50}}{(x^2 + 1)^{100}}$