

3.Arc length in polar coordinate

The length of arc in Cartesian coordinates when $y=f(x)$ is:

$$l = \int_a^b |v| dt$$

$$ds = \sqrt{dx^2 + dy^2} \rightarrow s = \int \sqrt{dx^2 + dy^2} ds$$

But in polar coordinate where we have

$$y = r \sin \theta \quad . x = r \cos \theta \quad . r = f(\theta)$$

$$\frac{dx}{d\theta} = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$\frac{ds}{d\theta} = \sqrt{\left(-r \sin \theta + \cos \theta \frac{dr}{d\theta}\right)^2 + \left(r \cos \theta + \sin \theta \frac{dr}{d\theta}\right)^2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \rightarrow ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

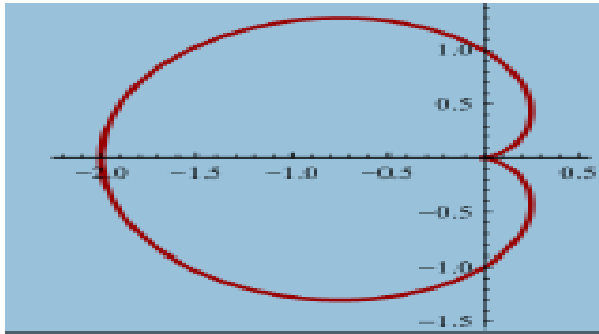
$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 1: Find length of the cardioid $r = a(1 - \cos \theta)$

from $\theta = 0$ to $\theta = 2\pi$

Solution /

$$\frac{dr}{d\theta} = a \sin \theta. \quad S = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$



$$= \int_0^{2\pi} \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos \theta + \cos^2 \theta) + a^2 \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} a\sqrt{2 - 2\cos \theta} d\theta = \int_0^{2\pi} \sqrt{2}a\sqrt{1 - \cos \theta} d\theta$$

$$[\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \leftrightarrow \sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta)]$$

$$= \int_0^{2\pi} \sqrt{2}a\sqrt{2\sin^2 \frac{\theta}{2}} d\theta = \int_0^{2\pi} \sqrt{2}a\sqrt{2}\sin \frac{\theta}{2} d\theta = 8a$$

H.W.: Find length of the cardioid $r = 2(1 + \cos \theta)$

10. Function of two or more variables

If (F) is a function of two independent variables we usually call the variables X and Y and the domain of (F) are a region in the xy-plane.

Domain:

Function	Domain	Range
$W = \sqrt{y - x}$	$y \geq x$	$w \geq 0$
$W = \frac{1}{xyz}$	$x, y \& z \neq 0$	$w \neq 0$
$W = \sin xy$	X, Y real number	$-1 \leq w \leq 1$
$w = \sqrt{x^2 + y^2 + z^2}$	All real number	$w \geq 0$

Limits:

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) , and write:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L$$

Properties of limits of functions of two variables:

The following rules hold if L , M , k are real numbers and:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0, y_0)} g(x, y) = M$$

$$1. \quad \lim_{(x,y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

$$2. \quad \lim_{(x,y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

$$3. \quad \lim_{(x,y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

$$4. \quad \lim_{(x,y) \rightarrow (x_0, y_0)} (kf(x, y)) = kL \quad (\text{any number of } k)$$

$$5. \quad \lim_{(x,y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad M \neq 0$$

Example:

$$\lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = \frac{3}{0 + 0 - 1} = -3$$

11. Partial derivatives of a function of two variables:-

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f(x, y) \quad (x_0, y_0)$$

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

Example 1: Find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ for $f(x, y) = x^2 + 2xy + y^2 + 5$

Solution//

$$\frac{\partial f}{\partial y} = 0 + 2x + 2y + 0$$

$$\frac{\partial f}{\partial x} = 2x + 2y + 0 + 0$$

Example 2: Find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ for $f(x, y) = y \sin xy$

Solution//

$$\frac{\partial f}{\partial y} = y \cos xy * x + \sin xy * 1 = xy \cos xy + \sin xy$$

$$\frac{\partial f}{\partial x} = y \cos xy * y = y^2 \cos xy$$

Example 3: Find $\frac{\partial f}{\partial z}$ for $f(x, y) = x \sin(y + 3z)$.

Solution //

$$\frac{\partial f}{\partial z} = 3x \cos (y + 3z)$$

H.W. Example: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x}$ for $f(x, y) = x^2 + y^2$

H.W. Example: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x}$ for $f(x, y) = \frac{\sin xy^2}{x^2+1}$

Notes:-

The higher order partial derivative:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{xy}$$

Example: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x, y) = x \cos y + ye^x$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x \cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x \cos y + ye^x) = -x \sin y + e^x$$

\therefore

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\cos y + ye^x) = ye^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (-x \sin y + e^x) = -x \cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (-x \sin y + e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (\cos y + ye^x) = -\sin y + e^x$$

H.W. Example 1: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial^2 f}{\partial x \partial y}$ for

$$f(x, y) = e^{x^2 y} + xy^3$$

H.W. Example 1: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial^2 f}{\partial x \partial y}$ for

$$f(x, y) = xy + \frac{e^y}{y^2 + 2}$$