15. Integration application/length of curve:-

Length of the curve (Length of arc)

$$L=\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx \quad with\ x-axis$$

$$\text{Or} \quad L=\int_c^d \sqrt{1+(\frac{dx}{dy})^2}\,dy \quad with\ y-axis$$

If x=f (t) , y =f (t) then $L=\int_{to}^{t1} \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$

Example 1: Find the length of the curve bounded by the curve $y=\frac{4\sqrt{2}}{3}x^{\frac{3}{2}}-1 \quad and\ 0 \le x \le 1$.

Solution //

$$\frac{dy}{dx}=\frac{4\sqrt{2}}{3}*\frac{3}{2}x^{\frac{1}{2}}=2\sqrt{2}x^{\frac{1}{2}}$$

$$L=\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx \quad \rightarrow L=\frac{2}{3}*\frac{1}{8}(1+8x)^{\frac{3}{2}}\Big|_0^1=\frac{13}{6}$$

Example: the curve $x=a\cos^3 t$ & $y=a\sin^3 t$

find the length of the curve $0 \le t \le \frac{\pi}{2}$.

Solution //

$$L=\int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t.\sin t)^2+(3a\sin^2 t.\cos t)^2}\,dt$$

$$L=\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4 t.\sin^2 t+9a^2\sin^4 t.\cos^2 t}\,dt$$

$$L=\int_0^{\frac{\pi}{2}} 3a\sqrt{\cos^2 t\,.\sin^2 t\,(\sin^2 t+\cos^2 t)}\ dt$$

$$L=\int_0^{\frac{\pi}{2}} 3a(\cos t\,.\sin t)dt=\int_0^{\frac{\pi}{2}} 3a(\sin 2t)dt=\frac{-3a}{4}(\cos 2t)\Big|_0^{\frac{\pi}{2}}=\frac{3a}{2}$$

16. Integration application/area of surface:-

Surface area: the area of surface swept out by revolving the curve about the axis.

1. Rotation with x-axis $[S=2\pi\int_a^b y\sqrt{1+(\frac{dy}{dx})^2}\ dx]$

2. Rotation with y-axis $[S=2\pi\int_c^d x\sqrt{1+(\frac{dx}{dy})^2}\ dy]$

3. If x=f (t) , y =f (t) $[S=2\pi\int_{t0}^{t1}\rho\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\ dt\ ]$

Where:-

ρ *is the distance from the axis of revolving to the element of arc length*

and expressses as function of t

Example: Find the area of surface obtained by revolving curve $y=\sqrt{x}$ ***with x − axis and*** $0\le x\le 2$

Solution//

$$S=2\pi\int_0^2 y\sqrt{1+(\frac{dy}{dx})^2}\ dx$$

$$\frac{dy}{dx}=\frac{1}{2}(x^{\frac{-1}{2}})\rightarrow(\frac{dy}{dx})^2=(\frac{1}{2}\left(x^{\frac{-1}{2}}\right))^2=\frac{1}{4x}$$

$$S=2\pi\int_0^2\sqrt{x}\sqrt{1+\frac{1}{4x}}\ dx=2\pi\int_0^2\sqrt{x}\sqrt{\frac{4x+1}{4x}}\ dx$$

$$=\frac{\pi}{6}(4x+1)^{\frac{3}{2}}\Big|_0^2=\frac{\pi}{6}[27-1]=\frac{13\pi}{3}$$

Example: Find the area of surface obtained by revolving curve $x=a\cos^3 t$ & $y=a\sin^3 t$ *with* $x-axis$ *and* $0\le x\le\frac{\pi}{2}$

Solution//

$$\frac{dx}{dt}=-3a\cos^2 t\,.\sin t\rightarrow(\frac{dx}{dt})^2=9a^2\cos^4 t\,.\sin^2 t$$

$$\frac{dy}{dt}=3a\sin^2 t\,.\cos t\rightarrow(\frac{dy}{dt})^2=9a\sin^4 t\,.\cos^2 t$$

$$\rho=\ y=a\sin^3 t$$

$$S=2\pi\int_{t0}^{t1}\rho\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=$$

$$2\pi\int_0^{\frac{\pi}{2}}a\sin^3 t\sqrt{9a^2\cos^4 t\,.\sin^2 t+9a\sin^4 t\,.\cos^2 t}\;dt=$$

$$2\pi\int_0^{\frac{\pi}{2}}a^2\sin^4 t\cos t\,dt=\frac{6\pi}{5}a^2\ unit^2$$