CHAPTER THREE

One-Dimensional, Steady-State Conduction

3.1 The Plane Wall

Consider the plane wall of Figure (3.1) where a direct application of Fourier's law may be made. Integration yields

$$q_x = -kA\frac{dT}{dx} = -\frac{kA}{\Delta x}\left(T_{s,2} - T_{s,1}\right) \qquad (3.1)$$

$$q_x = \frac{kA}{L}(T_{s1} - T_{s2}) = \frac{T_{s1} - T_{s2}}{R_{t,cond.}} \qquad (3.2)$$

Thermal resistance for conduction is

$$R_{t,cond.} = \frac{\Delta x}{kA} = \frac{L}{kA} \qquad (3.3)$$

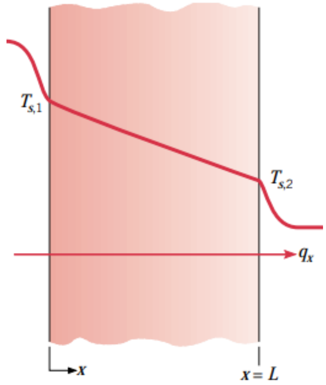


Figure (3.1) Heat Transfer through a Plane Wall.

The equivalent thermal circuit for the plane wall with convection surface conditions is shown in Figure (3.2). The heat transfer rate may be determined from separate consideration of each element in the network. Since (q_x) is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2A}$$

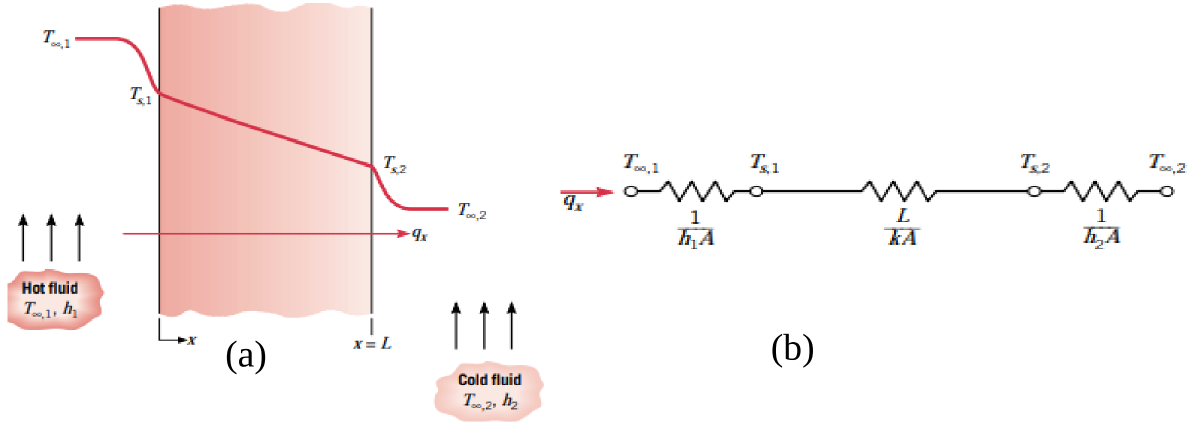


Figure (3.2) Heat Transfer through a Plane Wall. (a) Temperature Distribution. (b) Equivalent Thermal Circuit

The thermal resistance for convection is

$$R_{t,conv.} = \frac{1}{hA} \qquad (3.4)$$

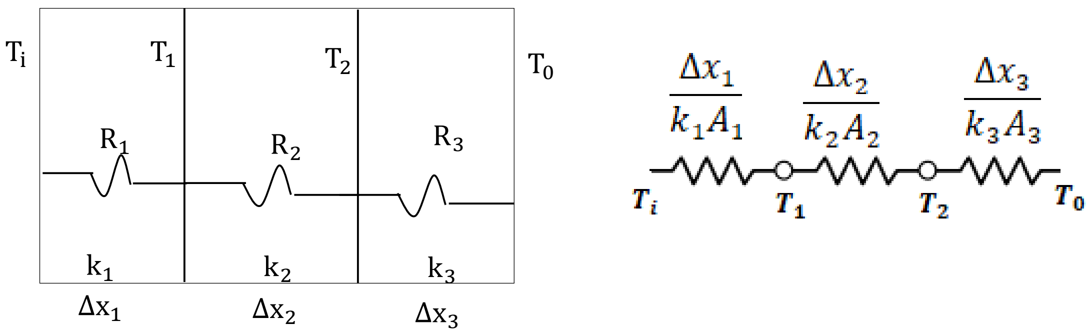
In terms of the overall temperature difference $(T_{\infty,1} - T_{\infty,2})$, and the total thermal resistance (R_{tot}) the heat transfer rate may also be expressed as

$$q_x = \frac{(T_{\infty,1} - T_{\infty,2})}{R_{tot}}$$

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

3.2 The Composite Wall

As shown in the figure below there are three walls in series with each other a situation which is similar to an electrical circuit consisting of three series resistors and battery across them.



The total resistance through the Composite wall is given by

$$\sum R_{tot} = R_1 + R_2 + R_3 = \frac{\Delta x_1}{k_1 A_1} + \frac{\Delta x_2}{k_2 A_2} + \frac{\Delta x_3}{k_3 A_3}$$

And gives a total heat flow (q) is

$$q = \frac{\Delta T_{overall}}{\sum R_{tot}} = \frac{T_i - T_0}{\Delta x_1/k_1 A_1 + \Delta x_2/k_2 A_2 + \Delta x_3/k_3 A_3} = UA\Delta T$$

Where U is the overall heat transfer coefficient

$$U = \frac{1}{\Delta x_1/k_1 + \Delta x_2/k_2 + \Delta x_3/k_3}$$

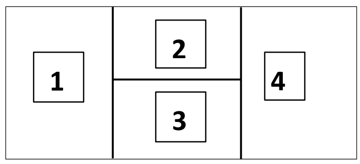
$$UA = \frac{1}{\sum R_{tot}} \qquad (3.5)$$

Noting that the heat flow across the first wall is equal to the second and third walls.

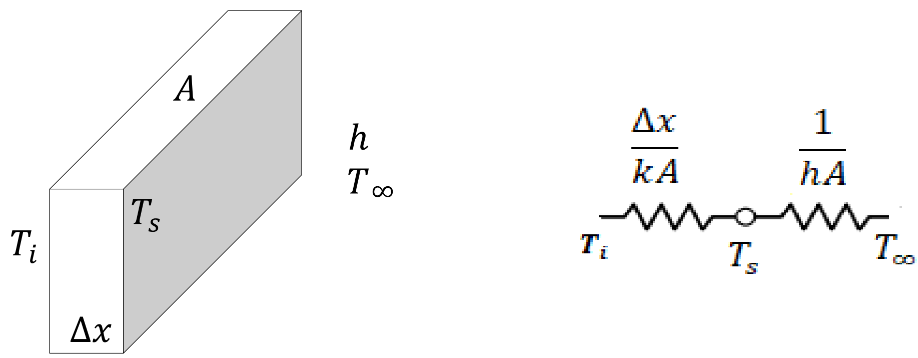
$$q = \frac{T_i - T_1}{\Delta x_1/k_1 A_1} = \frac{T_1 - T_2}{\Delta x_2/k_2 A_2} = \frac{T_2 - T_0}{\Delta x_3/k_3 A_3}$$

There are many other connect of thermal resistance. For example, the composite wall as shown below the overall resistance is:

$$\sum R_{th} = R_1 + \left[\frac{1}{R_2} + \frac{1}{R_3}\right]^{-1} + R_4$$



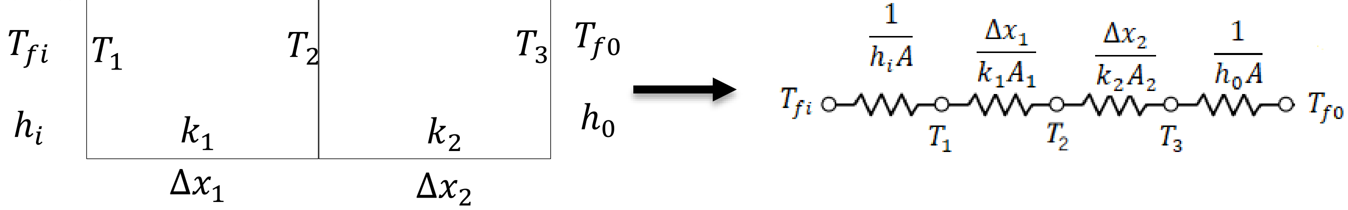
And when the case is a combination of conduction and convection as shown in the figure below the electrical analogy will be



and the heat flow (q) is

$$q = \frac{T_i - T_\infty}{\Delta x/kA + 1/hA}$$

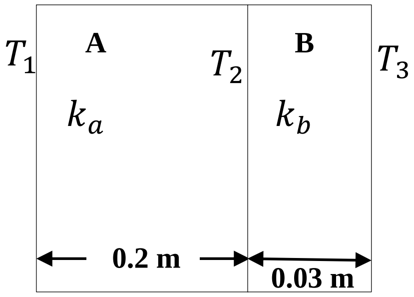
The general case of a combination of conduction and convection as shown in the figure below



$$q = \frac{\Delta T_{overall}}{\sum R_{tot}} = \frac{T_{fi} - T_{f0}}{\sum R_{tot}}, \qquad \sum R_{th} = \frac{1}{h_i A} + \frac{\Delta x_1}{k_1 A_1} + \frac{\Delta x_2}{k_2 A_2} + \frac{1}{h_0 A}$$

Example (3.1): A laboratory furnace wall is constructed of (0.2 m) thick fireclay with ($k_a = 1\ W/m.K$) this is covered on the outer surface with (0.03 m) thick layer of insulation material having ($k_b = 0.07\ W/m.K$) the furnace inner brick surface is at (1250 K) and the outer surface of the insulation is (310 K). Calculate the steady-state heat transfer rate through the wall in W/m^2, and determined the interfacial temperature (T_2) between the brick and insulation.

Solution:



$$\frac{q}{A} = \frac{T_1 - T_3}{\Delta x_a/k_a + \Delta x_b/k_b}$$

$$\frac{q}{A} = \frac{1250 - 310}{0.2/1 + 0.03/0.07} = 1495\ W/m^2$$

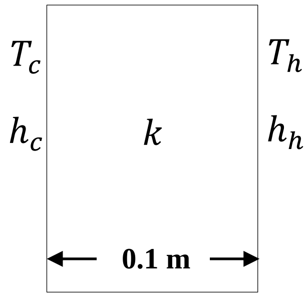
$$\frac{q}{A} = \frac{T_1 - T_2}{\Delta x_a/k_a}$$

$$1495 = \frac{1250 - T_2}{0.2/1}$$

$$T_2 = 951\ K$$

Example (3.2): A (0.1 m) thick brick wall (k=0.7 W/m. K) is exposed to a cold wind at (270 K) through a convection heat transfer coefficient of (40 $W/m^2.K$) on the other side is air at (330 K) with a natural convection heat transfer coefficient of (10 $W/m^2.K$). Calculate the rate of heat transfer per unit area.

Solution:



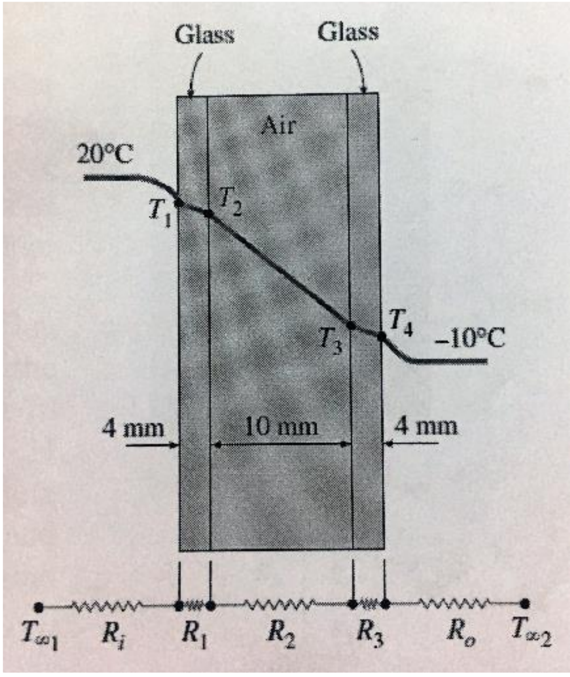
$$\frac{q}{A} = \frac{\Delta T}{1/h_h + \Delta x/k + 1/h_c}$$

$$\frac{q}{A} = \frac{330 - 270}{1/10 + 0.1/0.7 + 1/40}$$

$$\frac{q}{A} = 223.9\ W/m^2$$

Example (3.3): Consider a (0.8 m) high and (1.5 m) wide double pane window consisting of two (4 mm) thick layer of glass ($k = 0.78\ W/m.°C$) separated by (10 mm) wide stagnant air space ($k = 0.026\ W/m.°C$). Determined the steady rate of heat transfer through this double pane window and temperature of its inner surface for a day during which the room is maintained at (20 °C) while the temperature of outdoor is (-10 °C) take the convection heat transfer coefficient of the inner and outer surface of the window to be ($h_i = 10\ W/m^2.°C$), ($h_0 = 40\ W/m^2.°C$).

Solution:



$A = 0.8 * 1.5 = 1.2\ m^2$

$$R_i = \frac{1}{h_i A} = \frac{1}{10 * 1.2} = 0.083\ °C/W$$

$$R_1 = R_3 = R_{glass} = \frac{L_1}{k_1 A} = \frac{0.004}{0.78 * 1.2} = 0.00427\ °C/W$$

$$R_2 = R_{air} = \frac{L_2}{k_2 A} = \frac{0.01}{0.026 * 1.2} = 0.3205\ °C/W$$

$$R_0 = \frac{1}{h_0 A} = \frac{1}{40 * 1.2} = 0.02083\ °C/W$$

$$R_{total} = R_i + R_1 + R_2 + R_3 + R_0$$

$$R_{total} = 0.083 + 0.00427 + 0.3205 + 0.00427 + 0.02083$$

$$R_{total} = 0.4332\ °C/W$$

The steady rate of heat transfer through the double-pane window is:

$$q = \frac{\Delta T_{overall}}{R_{total}} = \frac{T_{\infty1} - T_{\infty2}}{R_{total}} = \frac{20 - (-10)}{0.4332} = 69.2\ W$$

The inner temperature is:

$$q = \frac{T_{\infty1} - T_1}{R_i}$$

$$69.2 = \frac{20 - T_1}{0.083}$$

$T_1 = 14.2\ °C$

3.3 Radial Systems

3.3.1 Cylinders

Consider a long cylinder of the inside radius (r_i), outside radius (r_0), and length (L), such as the one shown in Figure (3.12). We expose this cylinder to a temperature differential $(T_i - T_0)$. For a cylinder with length very large compared to diameter, it may be assumed that the heat flows only in a radial direction so that the only space coordinate needed to specify the system is (r_i). Again, Fourier's law is used by inserting the proper area relation. The area for heat flow in the cylindrical system is

$$A_r = 2\pi rL$$

$$q_r = -kA_r\frac{dT}{dr} = -2\pi Lk\frac{dT}{dr} \qquad (3.6)$$

The solution of the above equation is

$$q_r = 2\pi Lk\frac{(T_i - T_0)}{\ln(r_0/r_i)} = \frac{(T_i - T_0)}{R_{th}} \qquad (3.7)$$

and the thermal resistance in this case is

$$R_{th} = \frac{\ln(r_0/r_i)}{2\pi Lk} \qquad (3.8)$$

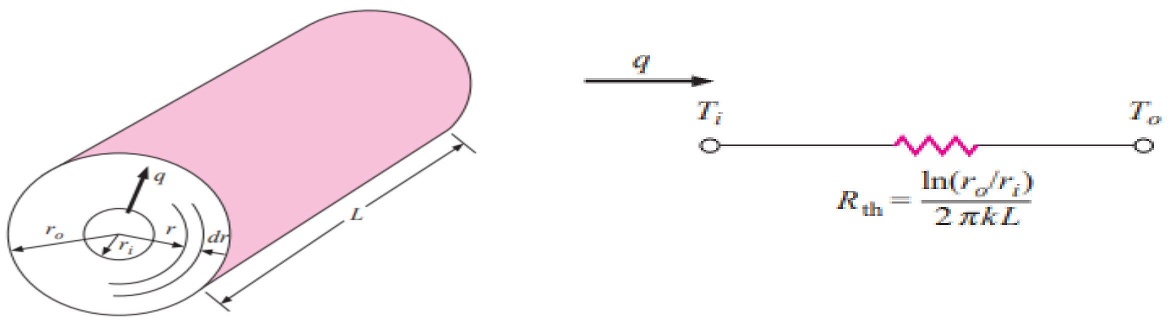


Figure (3.12) One Dimensional Heat Flow through a Hollow Cylinder and Electrical Analog.

The thermal resistance concept may be used for multiple layer cylindrical walls just as it was used for plane walls. For the three-layer system shown in Figure (3.13) the solution is

$$q_r = \frac{(T_i - T_0)}{R_{th}} = \frac{2\pi L(T_i - T_0)}{\ln(r_2/r_1)/k_A + \ln(r_3/r_2)/k_B + \ln(r_4/r_3)/k_C}$$

$$R_{th} = \frac{\ln r_2/r_1}{2\pi L k_A} + \frac{\ln r_3/r_2}{2\pi L k_B} + \frac{\ln r_4/r_3}{2\pi L k_C}$$

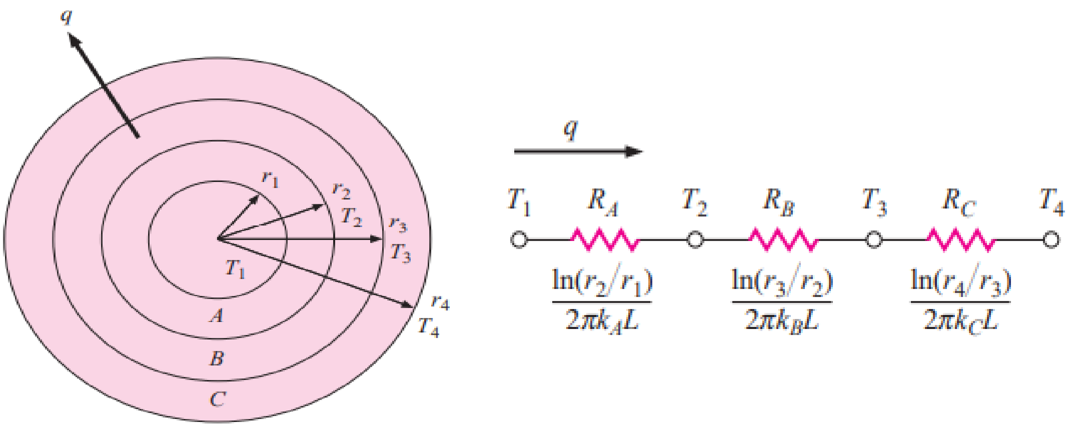


Figure (3.13) Thermal Resistance of Multiple Layer Cylindrical Walls

The hollow cylinder whose inner and outer surfaces are exposed to fluids at different temperatures Figure (3.14). The overall heat transfer would be expressed by

$$q = \frac{(T_{\infty,1} - T_{\infty,2})}{R_{tot}} = \frac{(T_{\infty,1} - T_{\infty,2})}{1/A_i h_i + \ln(r_0/r_i)/2\pi k L + 1/A_0 h_0}$$

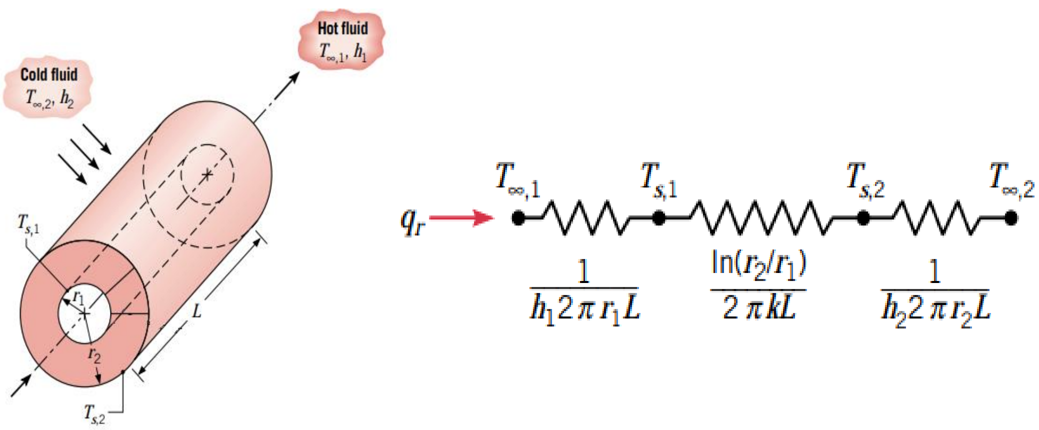
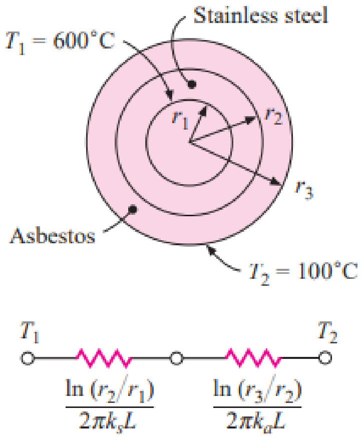


Figure (3.14) Hollow Cylinder with Convective Surface Conditions.

Example (3.4): A thick-walled tube of stainless steel (18% Cr, 8% Ni, k = 19 W/m · °C) with (2 cm) inner diameter and (4 cm) outer diameter is covered with a (3 cm) layer of asbestos insulation (k = 0.2 W/m. °C). If the inside wall temperature of the pipe is maintained at (600 °C) and the outside wall temperature is at (100 °C), calculate the heat loss per meter of length. Also calculate the tube insulation interface temperature.

Solution:



$$q/L = \frac{2\pi(T_1 - T_2)}{\ln(r_2/r_1)/k_s + \ln(r_3/r_2)/k_a}$$

$$q/L = \frac{2\pi(600 - 100)}{\ln(2/1)/19 + \ln(5/2)/0.2}$$

$$q/L = 680\ W/m$$

This heat flow may be used to calculate the interface temperature between the outside tube wall and the insulation. We have

$$q/L = \frac{(T_a - T_2)}{\ln(r_3/r_2)/2\pi k_a}$$

$$680 = \frac{2\pi(T_a - 100)}{\ln(5/2)/0.2}$$

$$T_a = 595.8\ °C$$

Example (3.5): Water flows at (50 °C) inside a (2.5 cm) inside diameter tube such that (h_i = 3500 W/m^2 · °C). The tube has a wall thickness of (0.8 mm) with thermal conductivity of (k=16 W/m. °C). The outside of the tube loses heat by free convection with (h_o = 7.6 W/m^2. °C). Calculate the heat loss per unit length to surrounding air at (20 °C).

Solution:

$$D_i = 0.025\ m \qquad \Rightarrow r_i = 0.0125m$$

$$D_0 = D_i + 2t = 0.025 + 2 * 0.0008 = 0.0266\ m \qquad \Rightarrow\ r_0 = 0.0133\ m$$

for unit length L = 1 m

$$A_i = \pi D_i L = \pi * 0.025 * 1 = 0.0785\ m^2$$

$A_0 = \pi D_0 L = \pi * 0.0266 * 1 = 0.0835\, m^2$

$$q = \frac{(T_i - T_0)}{R_{tot}} = \frac{(T_i - T_0)}{1/A_i h_i + \ln(r_0/r_i)/2\pi kL + 1/A_0 h_0}$$

$$q = \frac{(50 - 20)}{1/0.0785 * 3500 + \ln(0.0133/0.0125)/2\pi * 16 * 1 + 1/0.0835 * 7.6} = 19\, W$$

3.3.2 Spheres

Spherical systems may also be treated as one-dimensional when the temperature is a function of radius only. The heat flow is then

$$q_r = \frac{(T_i - T_0)}{R_{th}} = \frac{4\pi k(T_i - T_0)}{1/r_i - 1/r_0} \qquad (3.9)$$

$$R_{th} = \frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_0}\right) \qquad (3.10)$$

The hollow sphere, whose inner and outer surfaces are exposed to fluids at different temperatures Figure (3.15). The overall heat transfer would be expressed by

$$q = \frac{(T_h - T_c)}{R_{tot}} = \frac{(T_h - T_c)}{1/A_i h_h + (\frac{1}{r_1} - \frac{1}{r_2})/4\pi k_A + (\frac{1}{r_2} - \frac{1}{r_3})/4\pi k_B + 1/A_0 h_c}$$

$where\ A_i = 4\pi r_1^2$

and $A_0 = 4\pi r_3^2$

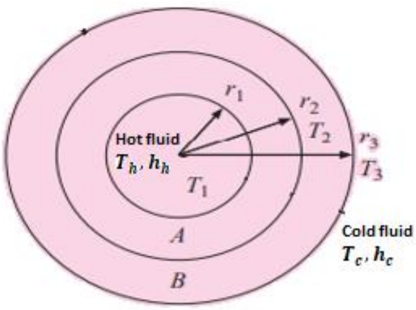
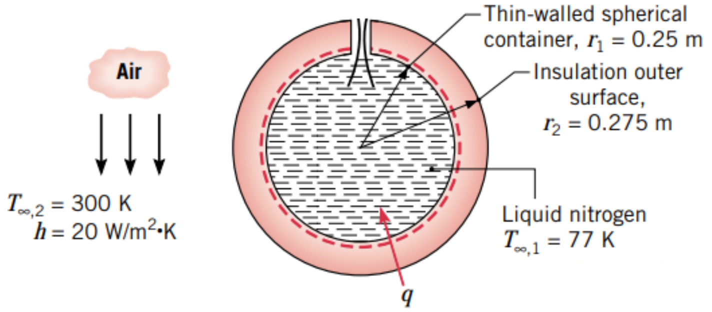
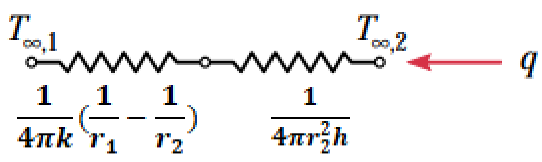


Figure (3.15) Hollow Sphere with Convective Surface Conditions.

Example (3.6): A spherical, thin-walled metallic container is used to store liquid nitrogen at (77 K). The container has a diameter of (0.5 m) and is covered with an evacuated, reflective insulation composed of silica powder with (k = 0.0017 W/m. K). The insulation is (25 mm) thick, and its outer surface is exposed to ambient air at (300 K). The convection coefficient is known to be (20 W/m^2. K). What is the rate of heat transfer to the liquid nitrogen?

Solution:





$$r_1 = \frac{D_1}{2} = \frac{0.5}{2} = 0.25\ m$$

$$r_2 = r_1 + t = 0.25 + 0.025 = 0.275\ m$$

$$q = \frac{(T_{\infty,2} - T_{\infty,1})}{R_{tot}}$$

$$q = \frac{(T_{\infty,2} - T_{\infty,1})}{(\frac{1}{r_1} - \frac{1}{r_2})/4\pi k + 1/4\pi r_2^2 h}$$

$$q = \frac{(300 - 77)}{\left(\frac{1}{0.25} - \frac{1}{0.275}\right)/4\pi * 0.0017 + 1/4\pi * 0.275^2 * 20}$$

$$q = \frac{223}{17.02 + 0.05}$$

$$q = 13.06\ W$$

3.4 Critical Thickness of Insulation

Let us consider a layer of insulation that might be installed around a circular pipe, as shown in Figure (3.16). The inner temperature of the insulation is fixed at T_i, and the outer surface is exposed to a convection environment at T_∞. From the thermal network, the heat transfer is

$$q = \frac{2\pi L(T_i - T_\infty)}{\ln(r_0/r_i)/k + 1/r_0 h_0} \qquad (3.11)$$

Now let us manipulate this expression to determine the outer radius of insulation (r_o) which will maximize the heat transfer. The maximization condition is

$$\frac{dq}{dr_0} = 0 = \frac{-2\pi L(T_i - T_\infty)(\frac{1}{kr_0} - \frac{1}{hr_0^2})}{[\ln(r_0/r_i)/k + 1/kr_0]^2} \qquad (3.12)$$

The result of the above equation (3.12) expresses the critical radius of insulation concept.

$$r_c = \frac{k}{h} \qquad (3.13)$$

If $r_0 < r_c$ then the heat transfer will be increased by adding more insulation.

If $r_0 > r_c$ then the heat transfer will be decreased by an increase in insulation thickness

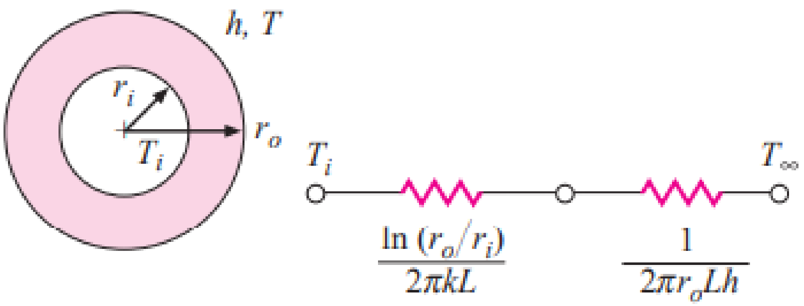


Figure (3.16) Critical Insulation Thickness.

The critical radius of insulation for the sphere is

$$r_c = \frac{2k}{h} \qquad (3.14)$$

Example (3.7): Calculate the critical radius of insulation for asbestos (k = 0.17 W/m · °C) surrounding a pipe and exposed to room air at (20 °C) with (h = 3.0 W/m^2 · °C). Calculate the heat loss from a (200 °C), (5 cm) diameter pipe when covered with the critical radius of insulation and without insulation.

Solution:

$$r_c = \frac{k}{h} = \frac{0.17}{3} = 0.0567\, m$$

$$r_i = D_i/2 = 0.05/2 = 0.025\, m$$

With insolation

$$q/L = \frac{2\pi(T_i - T_\infty)}{\ln(r_c/r_i)/k + 1/r_c h_0}$$

$$q/L = \frac{2\pi(200 - 20)}{\ln{(0.0567/0.025)}/0.017 + 1/0.0567 * 3}$$

$$q/L = 105.7\, W/m$$

Without insulation

$$q/L = 2\pi r_i h(T_i - T_\infty) = 2\pi * 0.025 * 3 * (200 - 20)$$

$$q/L = 84.8\, W/m$$

3.5 Heat Transfer from Extended Surfaces

The term extended surface is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid. The most frequent application is one in which an extended surface is termed a fin, which is used specifically to enhance heat transfer between a solid and an adjoining fluid. Different fin configurations are illustrated in Figure (3.17).

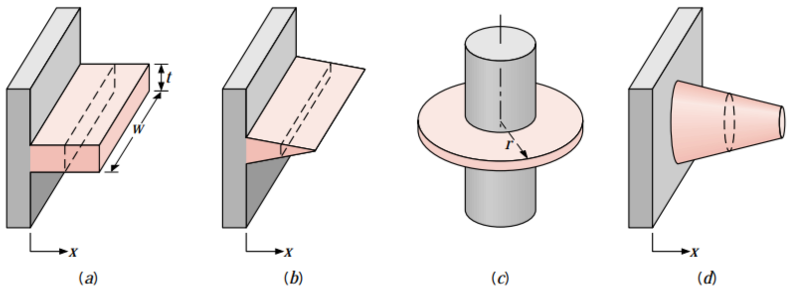


Figure (3.17) Fin Configurations. (a) Straight Fin of Uniform Cross Section. (b) Straight Fin of Non-Uniform Cross Section. (c) Annular Fin. (d) Pin Fin.

Consider the one-dimensional fin exposed to a surrounding fluid at a temperature (T_∞) as shown in Figure (3.18). We approach the problem by making an energy balance on an element of the fin of a thickness (dx) as shown in the figure. Applying the conservation of energy required to the differential element of Figure (3.18), we obtain

$$q_x = q_{x+dx} + dq_{conv} \tag{3.15}$$

$$q_x = -kA_c \frac{dT}{dx} \tag{3.16}$$

where (A_c) is the cross-sectional area, which may vary with (x). Since the conduction heat rate at ($x + dx$) may be expressed as

$$q_{x+dx} = q_x + \frac{dq_x}{dx}dx \tag{3.17}$$

$$q_{x+dx} = -kA_c \frac{dT}{dx} - k\frac{d}{dx}\left(A_c \frac{dT}{dx}\right)dx \tag{3.18}$$

$$dq_{conv} = h dA_s (T - T_\infty) \tag{3.19}$$

Where dA_s is the surface area of the differential element.

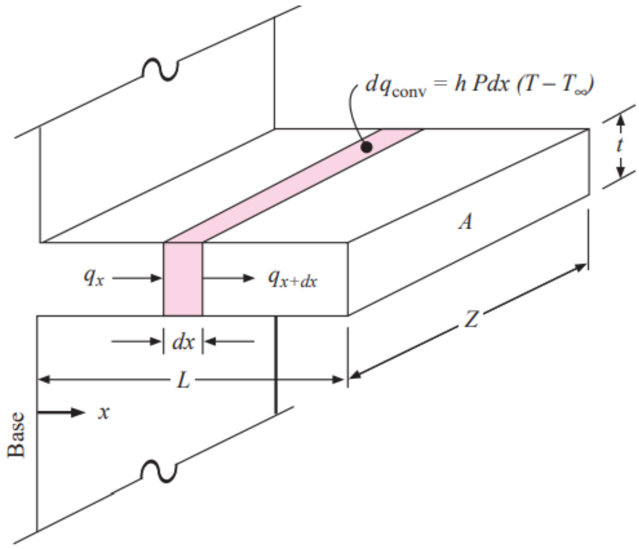


Figure (3.18) Sketch Illustrating One Dimensional Conduction and Convection through a Rectangular Fin.

Substituting the foregoing rate equations into the energy balance, Eq. (3.15), we obtain

$$-kA_c\frac{dT}{dx} = -kA_c\frac{dT}{dx} - k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx + hdA_s(T - T_\infty) \qquad (3.20)$$

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0 \qquad (3.21)$$

The general equation of heat transfer in a fin is

$$A_c\frac{d^2T}{dx^2} + \frac{dT}{dx}\frac{dA_c}{dx} - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0 \qquad (3.22)$$

For constant cross-sectional area A_c and $dA_s = Pdx$, where A_s is the surface area measured from the base to x and (P) is the fin perimeter. So that Eq. (3.22) reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad (3.23)$$

Let $(T - T_\infty) = \theta(x)$

$$\therefore \frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (3.24)$$

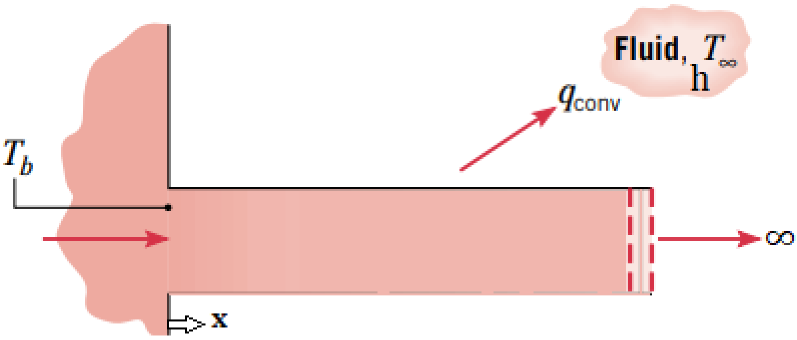
$$where \quad m = \sqrt{\frac{hP}{kA_c}}$$

The general solution for Eq. (3.24) may be written

$$\theta = C_1e^{mx} + C_2e^{-mx} \qquad (3.25)$$

The boundary condition depends on the physical situation. Several cases may be considered:

CASE (1): The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.



$$\theta = C_1 e^{mx} + C_2 e^{-mx} \qquad (1)$$

B.C 1: $at\ x = 0 \qquad T = T_b$

$T - T_\infty = T_b - T_\infty = \theta_b$ sub. In Eq. (1)

$$\theta_b = C_1 e^{m*0} + C_2 e^{-m*0}$$

$$\theta_b = C_1 + C_2 \qquad (2)$$

B.C 2: $at\ x = \infty \qquad T = T_\infty$

$T - T_\infty = T_\infty - T_\infty = 0 = \theta$ sub. in Eq. (1)

$$0 = C_1 e^{m*\infty} + C_2 e^{-m*\infty} \qquad where \quad e^{-\infty} = 0 \quad and \quad e^{\infty} = \infty$$

$0 = C_1 + 0 \qquad \therefore C_1 = 0$ sub. in Eq. (2)

$\theta_b = 0 + C_2 \qquad \therefore C_2 = \theta_b$ sub. in Eq. (1)

$$\theta = \theta_b e^{-mx}$$

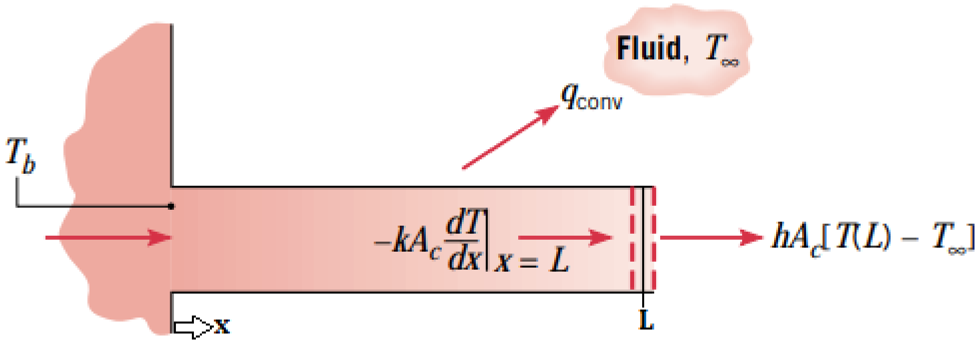
$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = e^{-mx} \qquad (3.26)$$

$$q = -kA \left.\frac{dT}{dx}\right]_{x=0} = -kA \left.\frac{d\theta}{dx}\right]_{x=0}$$

$$q = -kA\theta_b(-m)e^{-mx}]_{x=0} = kAm\theta_b$$

$$q = kA\sqrt{\frac{hP}{kA}}\theta_b = \sqrt{kAhP}\theta_b \qquad (3.27)$$

CASE (2): The fin is of finite length and loses heat by convection from its end.



$\theta = C_1 e^{mx} + C_2 e^{-mx}$ (1)

B.C **1:** $at\ x = 0 \quad T = T_b$

$T - T_\infty = T_b - T_\infty = \theta_b$ sub. in Eq. (1)

$\theta_b = C_1 e^{m*0} + C_2 e^{-m*0}$

$\theta_b = C_1 + C_2$ (2)

B.C 2: $at\ x = \mathrm{L} \quad q_{cond} = q_{conv}$

$$-kA \frac{d\theta}{dx}\bigg]_{x=L} = hA(T - T_\infty)_{x=L}$$

$-k(mC_1 e^{mL} - mC_2 e^{-mL}) = h(C_1 e^{mL} + C_2 e^{-mL})$ Divided by $(-km)$

$$C_1 e^{mL}\left(1 + \frac{h}{km}\right) = C_2 e^{-mL}\left(1 - \frac{h}{km}\right)$$

$C_1 = \frac{C_2 e^{-mL}\left(1 - \frac{h}{km}\right)}{e^{mL}\left(1 + \frac{h}{km}\right)}$ (3) sub. in Eq. (2)

$$\theta_b = \frac{C_2 e^{-mL}\left(1 - \frac{h}{km}\right)}{e^{mL}\left(1 + \frac{h}{km}\right)} + C_2$$

$C_2 = \frac{e^{mL}\left(1 + \frac{h}{km}\right)\theta_b}{e^{mL}\left(1 + \frac{h}{km}\right) + e^{-mL}\left(1 - \frac{h}{km}\right)}$ sub. in Eq. (3)

$$C_1=\frac{e^{-mL}(1-\frac{h}{km})e^{mL}\left(1+\frac{h}{km}\right)\theta_b}{e^{mL}\left(1+\frac{h}{km}\right)\left[e^{mL}\left(1+\frac{h}{km}\right)+e^{-mL}\left(1-\frac{h}{km}\right)\right]}$$

$$C_1=\frac{e^{-mL}(1-\frac{h}{km})\theta_b}{e^{mL}\left(1+\frac{h}{km}\right)+e^{-mL}\left(1-\frac{h}{km}\right)}$$

Sub. C_1 and C_2 in Eq. (11)

$$\theta=\frac{e^{-mL}(1-\frac{h}{km})\theta_b}{e^{mL}\left(1+\frac{h}{km}\right)+e^{-mL}\left(1-\frac{h}{km}\right)}e^{mx}+\frac{e^{mL}\left(1+\frac{h}{km}\right)\theta_b}{e^{mL}\left(1+\frac{h}{km}\right)+e^{-mL}\left(1-\frac{h}{km}\right)}e^{-mx}$$

$$\sinh x=\frac{e^{x}-e^{-x}}{2}\qquad and \qquad \cosh x=\frac{e^{x}+e^{-x}}{2}$$

$$\frac{\theta}{\theta_b}=\frac{e^{-m(L-x)}\left(1-\frac{h}{km}\right)+e^{m(L-x)}\left(1+\frac{h}{km}\right)}{e^{mL}\left(1+\frac{h}{km}\right)+e^{-mL}\left(1-\frac{h}{km}\right)}*\frac{2}{2}$$

$$\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)+\frac{h}{km}\sinh m(L-x)}{\cosh(mL)+\frac{h}{km}\sinh(mL)} \tag{3.28}$$

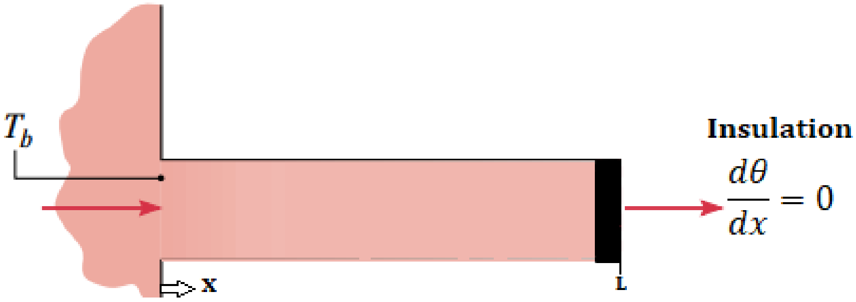
$$q=-kA\left.\frac{dT}{dx}\right]_{x=0}=-kA\left.\frac{d\theta}{dx}\right]_{x=0}$$

$$q=-kA*\left.\frac{\theta_b\left[-m\sinh m(L-x)+\frac{h}{km}*-m\cosh m(L-x)\right]}{\cosh(mL)+\frac{h}{km}\sinh(mL)}\right]_{x=0}$$

$$q=\sqrt{kA_chP}\,\theta_b\left[\frac{\sinh mL+\frac{h}{km}\cosh mL}{\cosh(mL)+\frac{h}{km}\sinh(mL)}\right] \tag{3.29}$$

CASE (3): The end of the fin is insulated so that

$$\frac{d\theta}{dx} = 0 \text{ at } x = L$$



$\theta = C_1 e^{mx} + C_2 e^{-mx}$ (1)

B.C 1: $at\ x = 0 \qquad T = T_b$

$T - T_\infty = T_b - T_\infty = \theta_b$ sub. in Eq. (1)

$\theta_b = C_1 e^{m*0} + C_2 e^{-m*0}$

$\theta_b = C_1 + C_2$ (2)

B. C 2: $at\ x = L \qquad \frac{d\theta}{dx} = 0$

$$\frac{d\theta}{dx} = 0 = mC_1 e^{mL} - mC_2 e^{-mL}$$

$$C_1 = C_2 \frac{e^{-mL}}{e^{mL}} \quad \text{sub. in Eq. (2)}$$

$$\theta_b = C_2 \frac{e^{-mL}}{e^{mL}} + C_2$$

$$C_2 = \frac{\theta_b e^{mL}}{e^{mL} + e^{-mL}}$$

$$C_1 = \frac{\theta_b e^{-mL}}{e^{mL} + e^{-mL}}$$

Sub. C_1 and C_2 in Eq. (1)

$$\theta = \frac{\theta_b e^{-mL}}{e^{mL} + e^{-mL}} e^{mx} + \frac{\theta_b e^{mL}}{e^{mL} + e^{-mL}} e^{-mx}$$

$$\frac{\theta}{\theta_b}=\frac{e^{-m(L-x)}+e^{m(L-x)}}{e^{mL}+e^{-mL}}*\frac{2}{2}$$

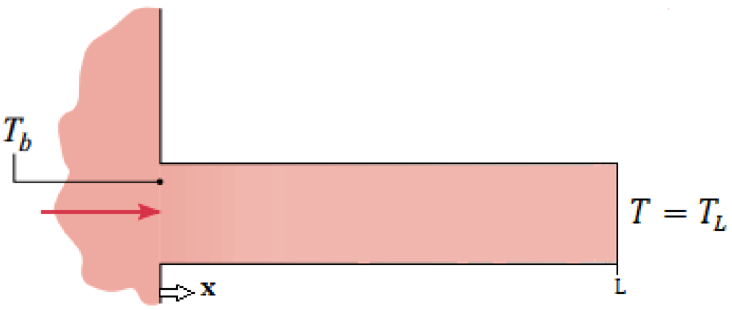
$$\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL} \qquad (3.30)$$

$$q=-kA\left.\frac{dT}{dx}\right]_{x=0}=-kA\left.\frac{d\theta}{dx}\right]_{x=0}$$

$$q=-kA\theta_{\mathrm{b}}\left.\frac{-m\sinh m(L-x)}{\cosh mL}\right]_{x=0}$$

$$q=\sqrt{kA_c hP}\,\theta_{\mathrm{b}}\frac{\sinh mL}{\cosh mL}=\sqrt{kAhP}\,\theta_{\mathrm{b}}\tanh mL \qquad (3.31)$$

CASE (4): The temperature at the end of the fin is fixed.



$$\theta=C_1e^{mx}+C_2e^{-mx} \qquad (1)$$

B.C 1: $at\ x=0 \qquad T=T_b$

$T-T_\infty=T_b-T_\infty=\theta_{\mathrm{b}}$ sub. in Eq. (1)

$$\theta_{\mathrm{b}}=C_1e^{m*0}+C_2e^{-m*0}$$

$$\theta_{\mathrm{b}}=C_1+C_2 \qquad (2)$$

B.C 2: $at\ x=\mathrm{L} \qquad T=T_\infty$

$T-T_\infty=T_{\mathrm{L}}-T_\infty=\theta_L$ sub. in Eq. (1)

$$\theta_L=C_1e^{mL}+C_2e^{-mL}$$

$$C_1=\frac{\theta_L-C_2e^{-mL}}{e^{mL}} \qquad \text{sub. in Eq. (2)}$$

$$\theta_b = \frac{\theta_L - C_2 e^{-mL}}{e^{mL}} + C_2$$

$$\therefore C_2 = \frac{\theta_b e^{mL} - \theta_L}{e^{mL} - e^{-mL}}$$

$$C_1 = \frac{\theta_L - \frac{\theta_b e^{mL} - \theta_L}{e^{mL} - e^{-mL}} e^{-mL}}{e^{mL}}$$

$$C_1 = \frac{(e^{mL}\theta_L - e^{-mL}\theta_L) + (-\theta_b e^{mL} e^{-mL} + \theta_L e^{-mL})}{(e^{mL} - e^{-mL})e^{mL}}$$

$$C_1 = \frac{\theta_L - \theta_b e^{-mL}}{e^{mL} - e^{-mL}}$$

Sub. C_1 and C_2 in Eq. (1)

$$\theta = \frac{\theta_L - \theta_b e^{-mL}}{e^{mL} - e^{-mL}} e^{mx} + \frac{\theta_b e^{mL} - \theta_L}{e^{mL} - e^{-mL}} e^{-mx}$$

$$\theta = \theta_b \left[\frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L - x)}{\sinh mL}\right] \quad (3.32)$$

$$q = -kA \left.\frac{dT}{dx}\right]_{x=0} = -kA \left.\frac{d\theta}{dx}\right]_{x=0}$$

$$q = -kA\theta_b \left[\left[\frac{\{(\theta_L/\theta_b)m \cosh mx + (-m)\cosh m(L - x)\}\sinh mL}{(\sinh mL)^2}\right]\right]_{x=0}$$

$$q = -kA\theta_b \left[\frac{\{(\theta_L/\theta_b)m - m\cosh mL\}}{\sinh mL}\right]$$

$$q = kA\theta_b m \left[\frac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL}\right]$$

$$q = \sqrt{kA_c hp}\,\theta_b \left[\frac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL}\right] \quad (3.33)$$

Note:

Rectangular fin	Pin fin
$\boldsymbol{P = 2W + 2t}$	$P = \pi D$
$\boldsymbol{A_c = Wt}$	$A_c = \frac{\pi}{4} D^2$

Example (3.8): A very long rod (5 mm) in diameter has one end maintained at (100 °C). The surface of the rod is exposed to ambient air at (25 °C) with a convection heat transfer coefficient of (100 W/m^2. K). Determine the temperature distributions along rods constructed from pure copper (k= 398 W/m. K). What are the corresponding heat losses from the rods?

Solution:

The temperature distributions are determined from CASE (1) Eq. (3.26), which may be expressed as

$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = e^{-mx} \qquad (3.26)$$

$$T = T_\infty + (T_b - T_\infty) e^{-mx}$$

$$q = \sqrt{kAhP}\,\theta_{\mathrm{b}}$$

$$q = \sqrt{k * \frac{\pi}{4} D^2 * h * \pi D}\,(T_b - T_\infty)$$

$$q = \sqrt{398 * \frac{\pi}{4} (0.005)^2 * 100 * \pi * 0.005}\,(100 - 25)$$

$$q = 8.3\ W$$

3.6 Fin Performance

1- Fin Effectiveness ($\boldsymbol{\varepsilon_f}$): It is defined as the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin. Therefore

$$\varepsilon_f = \frac{q_{fin}}{h A_c \theta_b} \qquad (3.34)$$

where A_c is the fin cross-sectional area at the base. The use of fins may rarely be justified unless $\varepsilon_f \geq 2$.

2- Fin Efficiency (η_f):

$$\eta_f = \frac{q_{fin}}{q_{max}} = \frac{q_{fin}}{hA_f\theta_b} \qquad (3.35)$$

Where A_f is the surface area of the fin.

For a straight fin of uniform cross-section and an adiabatic tip, yield

$$\eta_f = \frac{\tanh mL}{mL} \qquad (3.36)$$

Another method to find the fin efficiency (η_f) is:

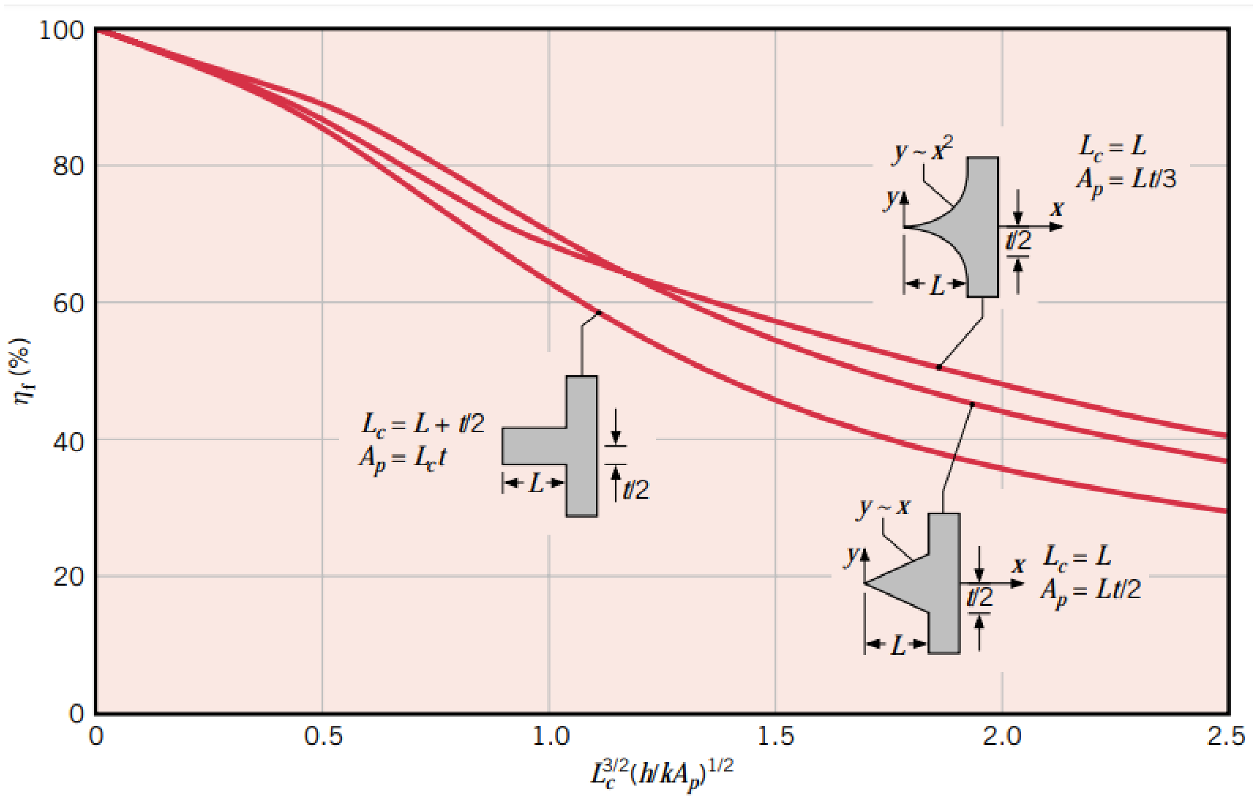


Figure (3.19) Efficiency of Straight Fins (Rectangular, Triangular, and Parabolic Profiles).

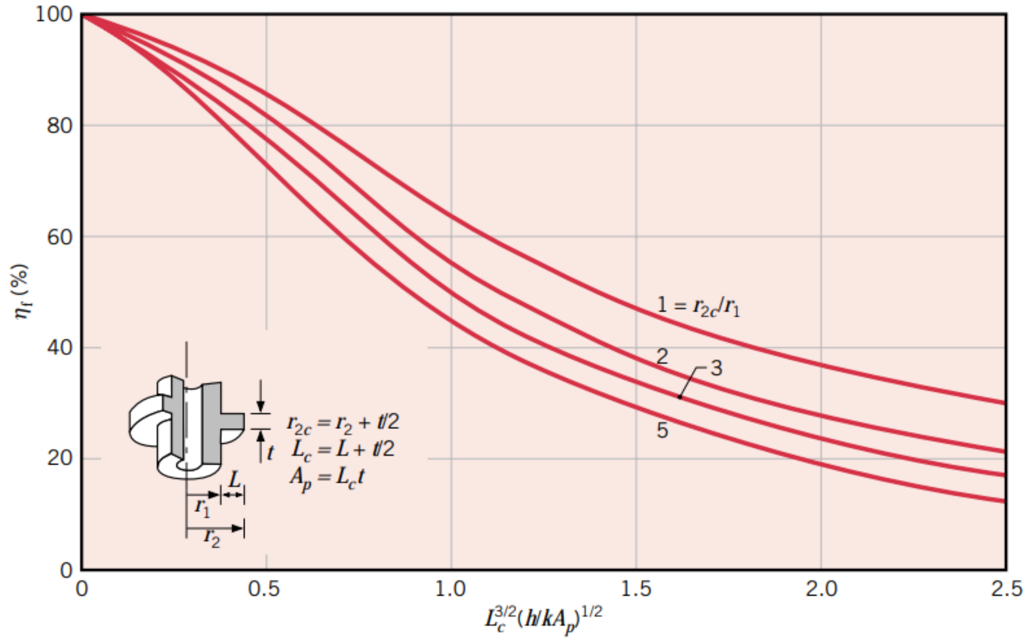
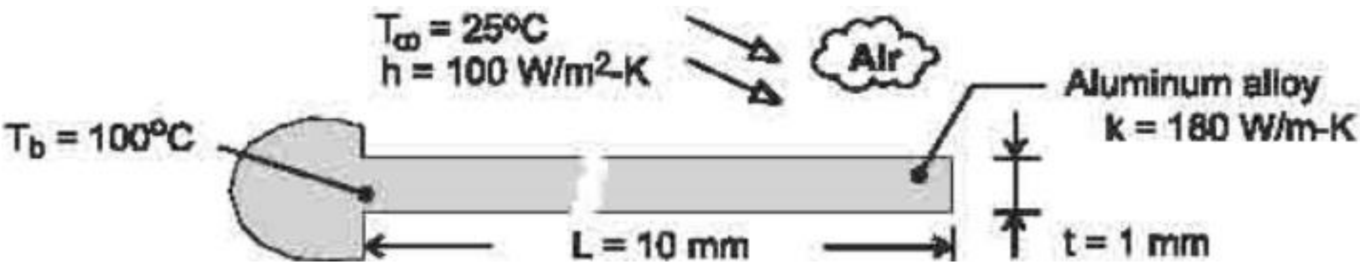


Figure (3.20) Efficiency of Annular Fins of Rectangular Profile.

Example (3.9): The extent to which the tip condition affects the thermal performance of a fin depends on the fin geometry and thermal conductivity, as well as the convection coefficient. Consider an alloyed aluminum (k =180 W/m. K) rectangular fin whose base temperature is $T_b = 100$ °C. The fin is exposed to a fluid of temperature $T_\infty = 25$°C, and a uniform convection coefficient of $(h = 100\ W/m^2.K)$ may be assumed for the fin surface. For a fin of length $(L = 10\ mm)$, thickness $(t = 1\ mm)$, and width $(w \gg t)$, determine the fin heat transfer rate per unit width, efficiency (η_f), effectiveness (ε_f) and tip temperature (T_L).

Solution:



$$q = \sqrt{kA_c hP}\theta_b \left[\frac{\sinh mL + \frac{h}{km}\cosh mL}{\cosh(mL) + \frac{h}{km}\sinh(mL)}\right]$$

$$A_c = Wt = 1 * 0.001 = 0.001\ m^2$$

$$P = 2W + 2t = 2(1 + 0.001) = 2.002\ m$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{100 * 2.002}{180 * 0.001}} = 33.35$$

$$q' = \sqrt{180 * 0.001 * 100 * 2.002}(100 - 25)\left[\frac{\sinh 33.35 * 0.01 + \frac{100}{180*33.35}\cosh 33.35 * 0.01}{\cosh(33.35 * 0.01) + \frac{100}{180*33.35}\sinh(33.35 * 0.01)}\right]$$

$$q' = 450.22 * \left[\frac{0.339 + 0.016 * 1.056}{1.056 + 0.016 * 0.339}\right]$$

$$q' = 150.96\ W/m$$

$$\varepsilon_f = \frac{q_{fin}}{hA_c\theta_b} = \frac{150.96}{100 * 0.001 * (100 - 25)}$$

$$\varepsilon_f = 20.13$$

$$A_f = 2L + t = 2 * 0.01 + 0.001 = 0.021\ m^2$$

$$\eta_f = \frac{q_{fin}}{q_{max}} = \frac{q_{fin}}{hA_f\theta_b} = \frac{150.96}{100 * 0.021 * (100 - 25)}$$

$$\eta_f = 95.85\ \%$$

$$\frac{\theta}{\theta_b} = \frac{T_L - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x) + \frac{h}{km}\sinh m(L - x)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$$

$$\frac{T_L - 25}{100 - 25} = \frac{\cosh m(L - L) + \frac{h}{km}\sinh m(L - L)}{\cosh(33.35 * 0.01) + \frac{100}{180*33.35}\sinh(33.35 * 0.01)}$$

$$\frac{T_L - 25}{100 - 25} = \frac{1 + 0}{1.056 + 0.016 * 0.339}$$

$$T_L = 95.69\ ^\circ C$$

Home Work (3):

1- The rear window of an automobile is defogged by passing warm air over its inner surface with (k = 1.4 W/m· K.). If the warm air is at ($T_{\infty,i} = 40\,°C$) and the corresponding convection coefficient is ($h_i = 30\,W/m^2.K$), what are the inner and outer surface temperatures of (4 mm) thick window glass if the outside ambient air temperature is ($T_{\infty,0} = -10$ °C) and the associated convection coefficient is ($h_0 = 65\,W/m^2.K$)?

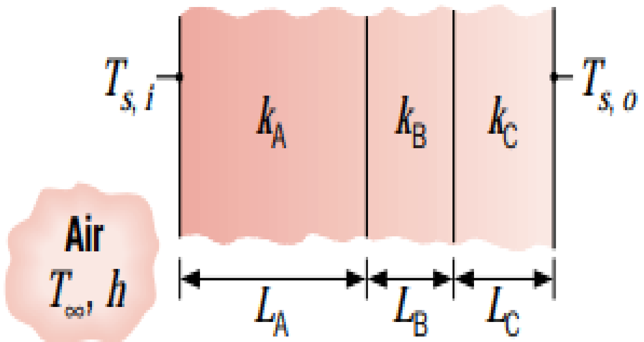
2- The walls of a refrigerator are typically constructed by sandwiching a layer of insulation between sheet metal panels. Consider a wall made from fiberglass insulation of thermal conductivity ($k_i = 0.046\,W/m.K$) and thickness ($L_i = 50\,mm$) and steel panels, each of thermal conductivity ($k_p = 60\,W/m.K$) and thickness ($L_p = 3\,mm$). If the wall separates refrigerated air at ($T_{\infty,i} = 4$ °C) from ambient air at ($T_{\infty,0} = 25$ °C), what is the heat gain per unit surface area? Coefficients associated with natural convection at the inner and outer surfaces may be approximated as ($h_i = h_0 = 5\,W/m^2.K$).

3- The wind chill, which is experienced on a cold, windy day, is related to increased heat transfer from exposed human skin to the surrounding atmosphere. Consider a layer of fatty tissue that is (3 mm) thick with (k = 0.2 W/m· K.) and whose interior surface is maintained at a temperature of (36 °C). On a calm day the convection heat transfer coefficient at the outer surface is (25 W/m^2. K), but with (30 km/h) winds it reaches (65 W/m^2. K). In both cases the ambient air temperature is (15 °C).

(a) What is the heat loss per unit area from the skin for the calm day and that for the windy day?

(b) What will be the skin outer surface temperature for the calm day? For the windy day?

4- The composite wall of an oven consists of three materials as shown below, two of which are of known thermal conductivity, ($k_A = 20\,W/m.K$) and ($k_C = 50\,W/m.K$), and known thickness, ($L_A = 0.30\,m$) and ($L_C = 0.15\,m$). The third material, B, which is sandwiched between materials A and C, is of known thickness, ($L_B = 0.15\,m$), but unknown thermal conductivity k_B. Under steady-state operating conditions, measurements reveal an outer surface temperature of ($T_{s,o} = 20\,°C$), an inner surface temperature of ($T_{s,i} = 600\,°C$), and an oven air temperature of $T_\infty = 800\,°C$. The inside convection coefficient (h) is known to be ($25\,W/m^2.K$). What is the value of (k_B)?



5- A stainless steel tube with ($k_{st} = 14.2\, W/m.K$) used to transport a chilled pharmaceutical has an inner diameter of (36 mm) and a wall thickness of (2 mm). The pharmaceutical and ambient air are at temperatures of (6 °C and 23 °C), respectively, while the corresponding inner and outer convection coefficients are (400 W/m^2. K) and (6 W/m^2. K), respectively.

(a) What is the heat gain per unit tube length?

(b) What is the heat gain per unit length if a 10 mm thick layer of calcium silicate insulation ($k_{ins} = 0.050\, W/m.K$) is applied to the tube?

6- Air flows at (120 °C) in a thin wall stainless steel tube with (h = 65 W/m^2. °C). The inside diameter of the tube is (2.5 cm) and the wall thickness is (0.4 mm). The tube is exposed to an environment with (h = 6.5 W/m^2. °C) and (T_∞ = 15 °C). The thermal conductivity of the steel is (k = 18 W/m. °C). Calculate the overall heat transfer coefficient and the heat loss per meter of length.

7- A hollow aluminum sphere of (k=230 W/m. K), with an electrical heater in the center, is used in tests to determine the thermal conductivity of insulating materials. The inner and outer radii of the sphere are (0.15) and (0.18 m), respectively, and testing is done under steady-state conditions with the inner surface of the aluminum maintained at (250 °C). In a particular test, a spherical shell of insulation is cast on the outer surface of the sphere to a thickness of (0.12 m). The system is in a room for which the air temperature is (20 °C) and the convection coefficient at the outer surface of the insulation is (30 W/m^2. K). If (80 W) is dissipated by the heater under steady-state conditions, what is the thermal conductivity of the insulation?

8- Aspherical vessel used as a reactor for producing pharmaceuticals has a (10 mm) thick stainless steel wall (k= 17 W/m. K) and an inner diameter of (l m). The exterior surface of the vessel is exposed to ambient air (T_∞ =25 °C) for which a convection coefficient of (6 W/m^2. K) may be assumed.

(a) During steady-state operation, an inner surface temperature of 50 °C is maintained by energy generated within the reactor. What is the heat loss from the vessel?

(b) If a (20 mm) thick layer of fiberglass insulation (k= 0.040 W/m. K) is applied to the exterior of the vessel and the rate of thermal energy generation is unchanged, what is the inner surface temperature of the vessel?

9- Calculate the critical radius of insulation for asbestos (k = 0.172 W/m. K) surrounding a pipe and exposed to room air at (300 K) with (h = 2.8 W/m^2. K). Calculate the heat loss from a (475 K), (60 mm) diameter pipe when covered with the critical radius of insulation and without insulation.

10- Find the amount of heat transferred through an iron fin of length (50 mm), width (100 mm) and thickness of (5 mm). Assume (k=210 W/m. °C) and (h=42 W/m^2. °C) for the material of the fin and the temperature at the base of the fin is (80 °C) and the surrounding temperature is (20 °C). Also, determine efficiency (η_f), effectiveness (ε_f) and tip temperature T_L. Assume the tip of the fin is insulation.

11- A straight fin of the rectangular profile has a thermal conductivity of (14 W/m. °C), the thickness of (2 mm), and length of (23 mm). The base of the fin is maintained at a temperature of (220 °C) while the fin is exposed to a convection environment at (23 °C) with (h = 25 W/m^2. °C). Calculate the fin heat transfer rate per meter of fin depth, efficiency (η_f) and effectiveness (ε_f).

12- A (40 mm) long, (2 mm) diameter pin fin is fabricated of an aluminum alloy (k=140 W/m. K).

(a) Determine the fin heat transfer rate and effectiveness (ε_f) for (T_b = 50 °C), (T_∞ = 25 °C), ($h = 1000\ W/m^2.K$), and an adiabatic tip condition.

(b) An engineer suggests that by holding the fin tip at a low temperature, the fin heat transfer rate can be increased. For $T(x = L) =$ 0 °C, determine the new fin heat transfer rate. Other conditions are as in part (a).

13- An experimental arrangement for measuring the thermal conductivity of solid materials involves the use of two long rods that are equivalent in every respect, except that one is fabricated from a standard material of known thermal conductivity (k_A) while the other is fabricated from the material whose thermal conductivity (k_B) is desired. Both rods are attached at one end to a heat source of fixed temperature (T_B), are exposed to a fluid of temperature (T_∞), and are instrumented with thermocouples to measure the temperature at a fixed distance (x_1) from the heat source. If the standard material is aluminum, with (k_A = 200 W/m. K), and measurements reveal values of (T_A = 75 °C) and (T_B = 60 °C) at (x_1)for (T_b = 100 °C) and (T_∞ = 25°C), what is the thermal conductivity (k_B) of the test material?