

Lumped Heat Capacity System

The lumped-heat-capacity method of analysis is used in which no temperature gradient exists. This means that the internal resistance of the body (conduction) is negligible in comparison with the external resistance (convection). i.e small h and large k.

For a hot body in a cold fluid, the energy balance is:

$$In - out = Accumulation$$

$$In = 0,$$

$$out = hA(T - T_{\infty})$$

$$Acc = mcp \frac{dT}{dt} \Rightarrow$$

$$-hA(T - T_{\infty}) = mcp \frac{dT}{dt} = \rho Vcp \frac{dT}{dt}$$

Where:

A = surface area for convection ($4\pi r^2$ for sphere).

V = volume ($\frac{4}{3} \pi r^3$ for sphere).

ρ = density of body.

cp = specific heat of the body.

$$\therefore \frac{dT}{dt} + \frac{hA}{\rho Vcp} (T - T_{\infty}) = 0$$

$$\text{Put } \theta = T - T_{\infty}, \quad \frac{1}{\tau} = \frac{hA}{\rho Vcp}, \quad \tau = \text{time constant}$$

$$\Rightarrow \frac{d\theta}{dt} + \frac{hA}{\rho Vcp} \theta = 0$$

The initial condition is

$$t = 0, \quad T = T_i, \quad \theta = \theta_i$$

The solution is

$$\frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}} = e^{-\left(\frac{hA}{\rho Vcp}\right)\tau}$$

The above system is called capacitance system.

Applicability of Lumped Heat Capacity System

The criteria to apply the capacitance assumption is

$$\frac{h(V/A)}{k} < 0.1$$

The ratio $V/A = s$ as a characteristic dimension of the solid, the dimensionless group is called the Biot number:

$$\frac{hs}{k} = \text{Biot number} = \text{Bi}$$

$s = 1/2$ (thick) = L for plate

$s = r/2$ for cylinder

$s = r/3$ for sphere

Steel Ball Cooling in Air

EXAMPLE 4-1

A steel ball [$c = 0.46 \text{ kJ/kg} \cdot ^\circ\text{C}$, $k = 35 \text{ W/m} \cdot ^\circ\text{C}$] 5.0 cm in diameter and initially at a uniform temperature of 450°C is suddenly placed in a controlled environment in which the temperature is maintained at 100°C . The convection heat-transfer coefficient is $10 \text{ W/m}^2 \cdot ^\circ\text{C}$. Calculate the time required for the ball to attain a temperature of 150°C .

■ Solution

We anticipate that the lumped-capacity method will apply because of the low value of h and high value of k . We can check by using Equation (4-6):

$$\frac{h(V/A)}{k} = \frac{(10)[(4/3)\pi(0.025)^3]}{4\pi(0.025)^2(35)} = 0.0023 < 0.1$$

$$T = 150^\circ\text{C} \quad \rho = 7800 \text{ kg/m}^3 \quad [486 \text{ lb}_m/\text{ft}^3]$$

$$T_\infty = 100^\circ\text{C} \quad h = 10 \text{ W/m}^2 \cdot ^\circ\text{C} \quad [1.76 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F}]$$

$$T_0 = 450^\circ\text{C} \quad c = 460 \text{ J/kg} \cdot ^\circ\text{C} \quad [0.11 \text{ Btu/lb}_m \cdot ^\circ\text{F}]$$

$$\frac{hA}{\rho c V} = \frac{(10)4\pi(0.025)^2}{(7800)(460)(4\pi/3)(0.025)^3} = 3.344 \times 10^{-4} \text{ s}^{-1}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau}$$

$$\frac{150 - 100}{450 - 100} = e^{-3.344 \times 10^{-4}\tau}$$

$$\tau = 5819 \text{ s} = 1.62 \text{ h}$$

Transit Heat Flow in a Semi-infinite Solid

Consider the semi-infinite solid shown in Figure below maintained at some initial temperature T_i . The surface temperature is suddenly lowered and maintained at a temperature T_0 . The temperature distribution in the solid can be obtained by solving the equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

The boundary and initial conditions are, with

1) Constant wall temperature

$$[1](x, 0) = T_i$$

$$[2](0, t) = T_0$$

$$[3](\infty, t) = T_i$$

The final solution with these boundary conditions is

$$\frac{T(x, t) - T_0}{T_i - T_0} = \operatorname{erf} \frac{x}{2\sqrt{\alpha t}}$$

α = thermal diffusivity = $k/\rho \cdot cp$ (m^2/s).

Where the Gauss error function is defined as

$$\operatorname{erf} = \frac{2}{\pi} \int_0^x e^{-\eta^2} d\eta$$

or

$$\operatorname{erf} \frac{x}{2\sqrt{\alpha \tau}} = \frac{2}{\sqrt{\pi}} \int^{x/2\sqrt{\alpha \tau}} e^{-\eta^2} d\eta$$

η is a dummy variable

$$\frac{T(x, \tau) - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}} \int^{x/2\sqrt{\alpha \tau}} e^{-\eta^2} d\eta$$

The heat flow at any x position may be obtained from

$$q_x = -kA \frac{\partial T}{\partial x}$$

$\frac{\partial T}{\partial x}$ can be obtained by differentiating the eq. above

$$\frac{T(x,t) - T_0}{T_i - T_0} = \operatorname{erf} \frac{x}{2\sqrt{\alpha t}}$$

Since

$$\frac{\partial}{\partial x} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} dx$$

$$\therefore \frac{\partial T}{\partial x} = (T_i - T_0) \frac{2}{\sqrt{\pi}} e^{-x^2/4\alpha t} \frac{\partial}{\partial x} \left(\frac{x}{2\sqrt{\alpha t}} \right)$$

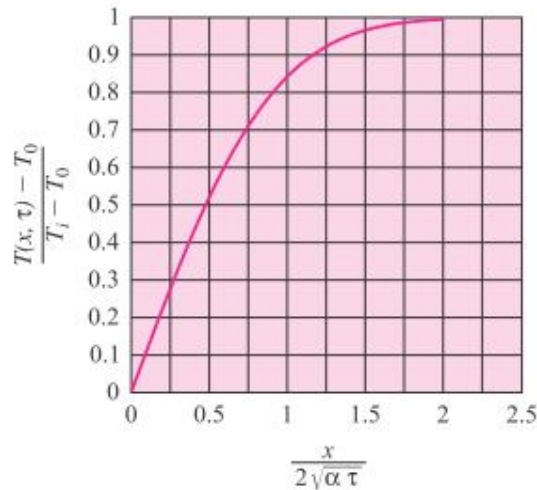
$$\therefore \frac{\partial T}{\partial x} = \frac{(T_i - T_0)}{\sqrt{\pi\alpha t}} e^{-x^2/4\alpha t} \Rightarrow$$

$$q_x = -kA \frac{(T_i - T_0)}{\sqrt{\pi\alpha t}} e^{-x^2/4\alpha t}$$

At the surface of solid ($x = 0$), the heat flux is

$$q_x = -kA \frac{(T_i - T_0)}{\sqrt{\pi\alpha t}}$$

A plot of the temperature distribution for the semi-infinite solid with constant wall temperature is given in Figure 4-4 below.



2) Constant heat flux

For the same uniform initial temperature distribution, we could suddenly expose the surface to a constant surface heat flux q_0/A . The initial and boundary conditions for this case are:

$$[1] (x, 0) = T_i$$

$$[2] \text{ at } x = 0 \quad \frac{q_0}{A} = -k \frac{\partial T}{\partial x} \Big|_{x=0} \quad \text{for } t > 0$$

$$[3] (\infty, t) = T_i$$

The final solution with these boundary conditions is

$$T - T_i = \frac{2q_0\sqrt{\alpha t/\pi}}{kA} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_0x}{kA} \left(1 - \operatorname{erf} \frac{x}{2\sqrt{\alpha t}}\right)$$

Semi-Infinite Solid with Sudden Change in Surface Conditions

EXAMPLE 4-2

A large block of steel [$k = 45 \text{ W/m} \cdot ^\circ\text{C}$, $\alpha = 1.4 \times 10^{-5} \text{ m}^2/\text{s}$] is initially at a uniform temperature of 35°C . The surface is exposed to a heat flux (*a*) by suddenly raising the surface temperature to 250°C and (*b*) through a constant surface heat flux of $3.2 \times 10^5 \text{ W/m}^2$. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

■ Solution

We can make use of the solutions for the semi-infinite solid given as Equations (4-8) and (4-13a). For case *a*,

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.025}{(2)[(1.4 \times 10^{-5})(30)]^{1/2}} = 0.61$$

The error function is determined from Appendix A as

$$\operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} = \operatorname{erf} 0.61 = 0.61164$$

We have $T_i = 35^\circ\text{C}$ and $T_0 = 250^\circ\text{C}$, so the temperature at $x = 2.5 \text{ cm}$ is determined from Equation (4-8) as

$$\begin{aligned} T(x, \tau) &= T_0 + (T_i - T_0) \operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} \\ &= 250 + (35 - 250)(0.61164) = 118.5^\circ\text{C} \end{aligned}$$

For the constant-heat-flux case *b*, we make use of Equation (4-13a). Since q_0/A is given as $3.2 \times 10^5 \text{ W/m}^2$, we can insert the numerical values to give

$$\begin{aligned} T(x, \tau) &= 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} e^{-(0.61)^2} \\ &\quad - \frac{(0.025)(3.2 \times 10^5)}{45} (1 - 0.61164) \\ &= 79.3^\circ\text{C} \quad x = 2.5 \text{ cm}, \tau = 30 \text{ s} \end{aligned}$$

For the constant-heat-flux case the *surface* temperature after 30 s would be evaluated with $x = 0$ in Equation (4-13a). Thus,

$$T(x=0) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} = 199.4^\circ\text{C}$$

3) Convection Boundary Conditions

For the semi-infinite-solid problem, the convection boundary condition is expressed by

Heat convected into surface = heat conducted into surface

$$[1] (x, 0) = T_i$$

$$[2] hA(T - T_\infty)_{x=0} = -k \frac{\partial T}{\partial x} \Big|_{x=0}$$

$$[3] (\infty, t) = T_i$$

The final solution with these boundary conditions is

$$\frac{T - T_i}{T_\infty - T_i} = \left(1 - \operatorname{erf} \frac{x}{2\sqrt{\alpha t}} \right) - \left[\exp \left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2} \right) \right] \times \left[1 - \operatorname{erf} \left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \right]$$

Where

T_i = initial temperature of solid

T_∞ = environment temperature

Example : A water pipe is buried 0.37m below ground in wet soil ($\alpha = 7.75 \times 10^{-7} \text{ m}^2/\text{s}$ and $k = 2.59 \text{ W/m.K}$). The soil is initially at a uniform temperature of 5°C for sudden application of a convective surface condition of wind with $h = 57 \text{ W/m}^2 \cdot ^\circ\text{C}$ and $T_\infty = -21^\circ\text{C}$, will the pipe be exposed to freezing temperature 0°C in a 10 hr period.

Solution

$$\frac{h\sqrt{\alpha t}}{k} = 3.68, \quad \frac{x}{2\sqrt{\alpha t}} = 1.11, \quad \operatorname{erf} \left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \approx 1.0$$

$$\frac{T - T_i}{T_\infty - T_i} = \left(1 - \operatorname{erf} \frac{x}{2\sqrt{\alpha t}} \right) = 1 - 0.89 \Rightarrow T = 2.14^\circ\text{C}$$