Partial Derivatives

Function of two or more variables:

Suppose D is a set of n – tuples of real numbers:

$$(x_1, x_2, \ldots\ldots, x_n)$$

A **real - valued function** f on D is a rule that assigns a unique (single) real number:

$$w = f(x_1, x_2, \ldots\ldots, x_n)$$

To each element in D . The set D is the function's **domain.** The set of w – values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n.

In the function $V = \pi r^2 h$, the dependent variable is V, the independent variables are r and h.

Example: Find value $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (3,0,4):

***Solution*:**

$$f(3,0,4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{9+0+16} = \sqrt{25} = 5$$

Limit of a function of two variables:

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_o, y_o), and write:

$$\lim_{(x,y)\to(x_o,y_o)} f(x, y) = L$$

If for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta$

Properties of limits of functions of two variables:

The following rules hold if $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{k}$ are real numbers and:

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_o,y_o)} g(x,y) = M$$

1. $$\lim_{(x,y)\to(x_o,y_o)} (f(x,y) + g(x,y)) = L + M$$

2. $$\lim_{(x,y)\to(x_o,y_o)} (f(x,y) - g(x,y)) = L - M$$

3. $$\lim_{(x,y)\to(x_o,y_o)} (f(x,y).g(x,y)) = L.M$$

4. $$\lim_{(x,y)\to(x_o,y_o)} (kf(x,y)) = kL$$ (any number of k)

5. $$\lim_{(x,y)\to(x_o,y_o)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$ $M \neq 0$

6. $$\lim_{(x,y)\to(x_o,y_o)} (f(x,y))^{r/s} = L^{r/s}$$ (where r and s are integers and $s \neq 0$)

Partial Derivatives:

Partial derivatives are the derivatives we get when we hold constant all but one of the independent variable in a function and differentiate with respect to that one.

Partial derivatives of a function of two variables:

The partial derivative of $f(x,y)$ with respect to *x* at the point (x_o, y_o) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o+h, y_o) - f(x_o, y_o)}{h}$$

Provided the limit exists

The partial derivative of $f(x,y)$ with respect to *y* at the point (x_o, y_o) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} = \left.\frac{d}{dy} f(x_o, y)\right|_{y=y_o} = \lim_{h\to 0}\frac{f(x_o, y_o+h) - f(x_o, y_o)}{h}$$

Provided the limit exists

$$f_x = \frac{\partial f}{\partial x} \quad , \quad f_y = \frac{\partial f}{\partial y}$$

functions of more than two variables:

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Second – order partial derivatives:

When we differential a function $f(x, y)$ twice, we produce its second – order derivative. These derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Partial derivatives of higher order:

Although we will deal mostly with first and second – order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third and fourth – order derivatives by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

***The chain Rule for functions of two variables*:**

The chain rule formula for function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

> If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(y))$ is a differentiable function of t and
>
> $$\frac{df}{dt} = f_x(x(t), y(t)).x'(t) + f_y(x(t), y(t)).y'(t)$$
>
> **or**
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> $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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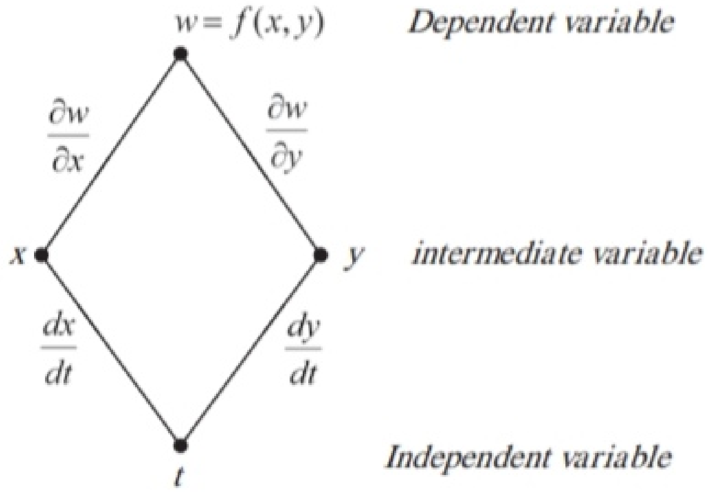
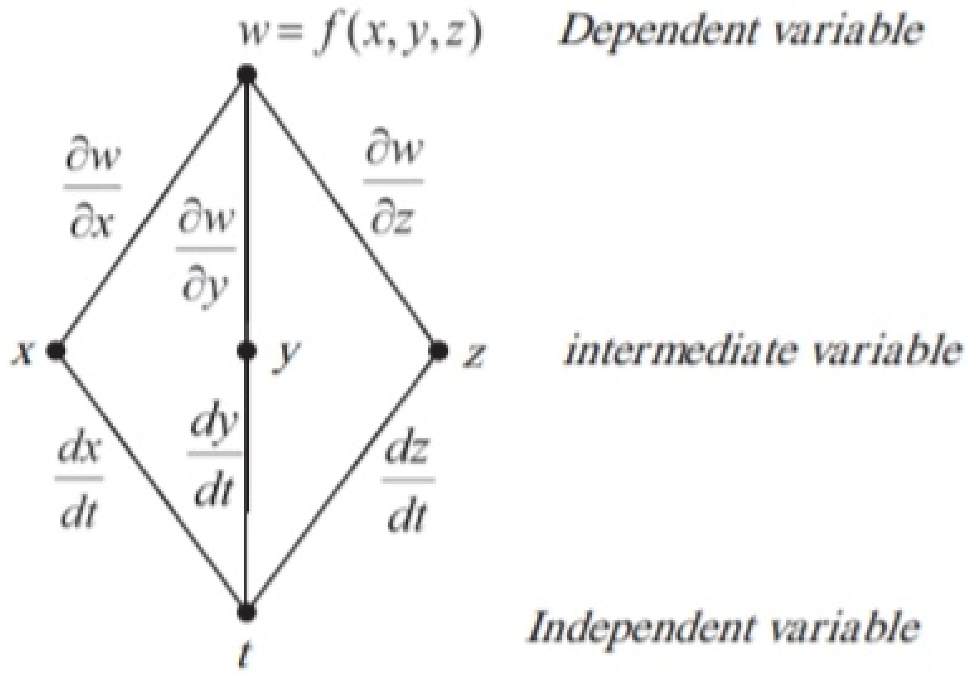


$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

The chain rule for function of three variables:

If $w = f(x, y, z)$ is differentiable and x , y and z are differentiable function of t , then w is a differentiable function of t and:

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$