كلية المستقبل الجامعة
قسم هندسة تقنيات البناء والانشاءات

Bending stress in beams

المرحلة الثانية

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3. BEAMS: STRAIN, STRESS, DEFLECTIONS

The beam, or flexural member, is frequently encountered in structures and machines, and its elementary stress analysis constitutes one of the more interesting facets of mechanics of materials. A beam is a member subjected to loads applied transverse to the long dimension, causing the member to bend. For example, a simply-supported beam loaded at its third-points will deform into the exaggerated bent shape shown in Fig. 3.1

Before proceeding with a more detailed discussion of the stress analysis of beams, it is useful to classify some of the various types of beams and loadings encountered in practice. Beams are frequently classified on the basis of supports or reactions. A beam supported by pins, rollers, or smooth surfaces at the ends is called a simple beam. A simple support will develop a reaction normal to the beam, but will not produce a moment at the reaction. If either, or both ends of a beam projects beyond the supports, it is called a simple beam with overhang. A beam with more than simple supports is a continuous beam. Figures 3.2a, 3.2b, and 3.2c show respectively, a simple beam, a beam with overhang, and a continuous beam. A cantilever beam is one in which one end is built into a wall or other support so that the built-in end cannot move transversely or rotate. The built-in end is said to be fixed if no rotation occurs and restrained if a limited amount of rotation occurs. The supports shown in Fig. 3.2d, 3.2e and 3.2f represent a cantilever beam, a beam fixed (or restrained) at the left end and simply supported near the other end (which has an overhang) and a beam fixed (or restrained) at both ends, respectively.

Cantilever beams and simple beams have two reactions (two forces or one force and a couple) and these reactions can be obtained from a free-body diagram of the beam by applying the equations of equilibrium. Such beams are said to be statically determinate since the reactions can be obtained from the equations of equilibrium. Continuous and other beams with only transverse loads, with more than two reaction components are called statically indeterminate since there are not enough equations of equilibrium to determine the reactions.

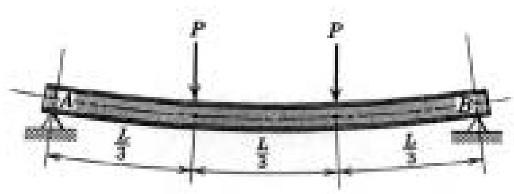

Figure 3.1 Example of a bent beam (loaded at its third points)

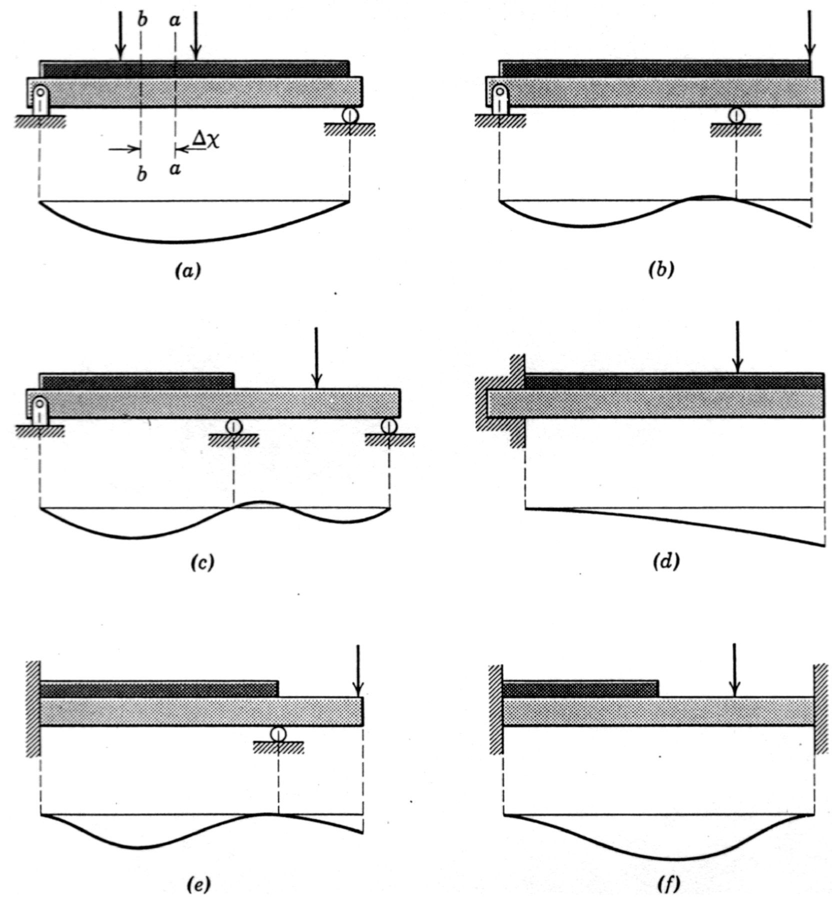

Figure 3.2 Various types of beams and their deflected shapes: a) simple beam, b) beam with overhang, c) continuous beam, d) a cantilever beam, e) a beam fixed (or restrained) at the left end and simply supported near the other end (which has an overhang), f) beam fixed (or restrained) at both ends.

Examining the deflection shape of Fig. 3.2a, it is possible to observe that longitudinal elements of the beam near the bottom are stretched and those near the top are compressed, thus indicating the simultaneous existence of both tensile and compressive stresses on transverse planes. These stresses are designated fibre or flexural stresses. A free body diagram of the portion of the beam between the left end and plane a-a is shown in Fig. 3.3. A study of this section diagram reveals that a transverse force V_r and a couple M_r at the cut section and a force, R, (a reaction) at the left support are needed to maintain equilibrium. The force V_r is the resultant of the shearing stresses at the section (on plane a-a) and is called the resisting shear and the moment, M_r, is the resultant of the normal stresses at the section and is called the resisting moment.

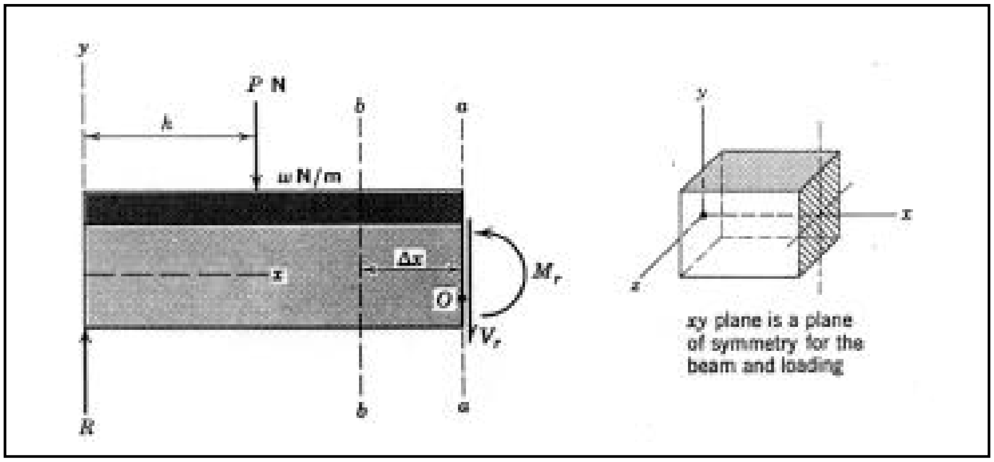

Figure 3.3 Section of simply supported beam.

The magnitudes and senses of V_r and M_r may be obtained form the equations of equilibrium $\sum F_y = 0$ and $\sum M_O = 0$ where O is any axis perpendicular to plane xy (the reaction R must be evaluated first from the free body of the entire beam). For the present the shearing stresses will be ignored while the normal stresses are studied. The magnitude of the normal stresses can be computed if M_r is known and also if the law of variation of normal stresses on the plane a-a is known. Figure 3.4 shows an initially straight beam deformed into a bent beam.

A segment of the bent beam in Fig. 3.3 is shown in Fig. 3.5 with the distortion highly exaggerated. The following assumptions are now made

i) Plane sections before bending, remain plane after bending as shown in Fig. 3.4 (Note that for this to be strictly true, it is necessary that the beam be bent only with couples (i.e., no shear on transverse planes), that the beam must be proportioned such that it will not buckle and that the applied loads are such that no twisting occurs.

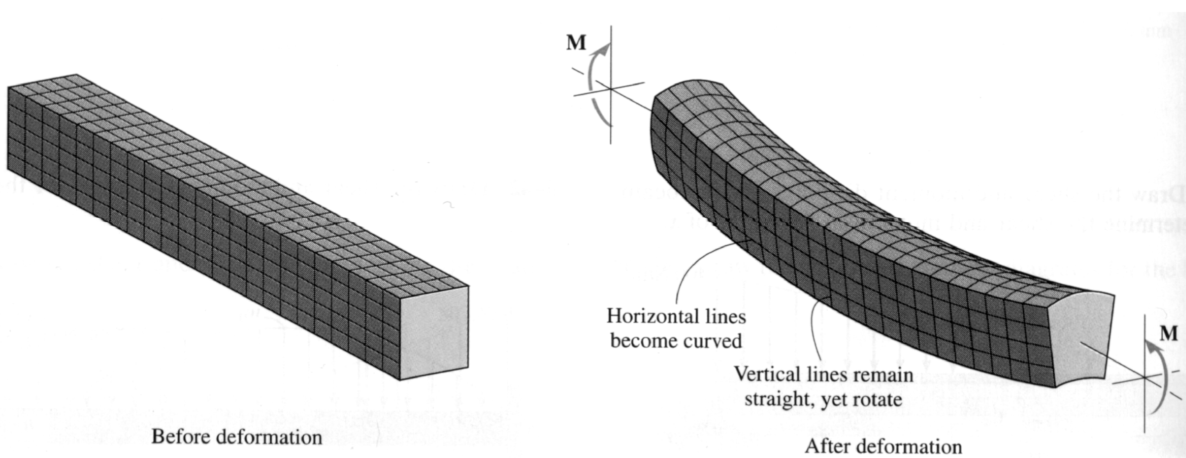

Figure 3.4 Initially straight beam and the deformed bent beam

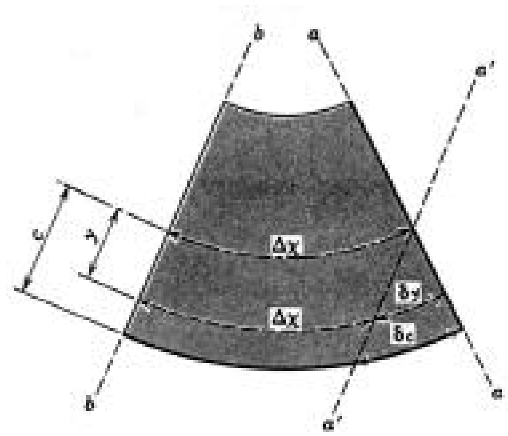

Figure 3.5 Distorted section of bent beam

ii) All longitudinal elements have the same length such the beam is initially straight and has a constant cross section.
iii) A neutral surface is a curved surface formed by elements some distance, c, from the outer fibre of the beam on which no change in length occurs. The intersection of the neutral surface with the any cross section is the neutral axis of the section.

Strain

Although strain is not usually required for engineering evaluations (for example, failure theories), it is used in the development of bending relations. Referring to Fig. 3.5, the following relation is observed:

$$\frac{\delta_y}{y} = \frac{\delta_c}{c} \tag{3.1}$$

where δ_y is the deformation at distance y from the neutral axis and δ_c is the deformation at the outer fibre which is distance c from the neutral axis. From Eq. 3.1, the relation for the deformation at distance y from the neutral axis is shown to be proportional to the deformation at the outer fibre:

$$\delta_y = \frac{\delta_c}{c} y \tag{3.2}$$

Since all elements have the same initial length, Δx, the strain at any element can be determined by dividing the deformation by the length of the element such that:

$$\frac{\delta_y}{\Delta x} = \frac{y}{c}\frac{\delta_c}{\Delta x} \qquad \varepsilon = \frac{y}{c}\varepsilon_c \tag{3.3}$$

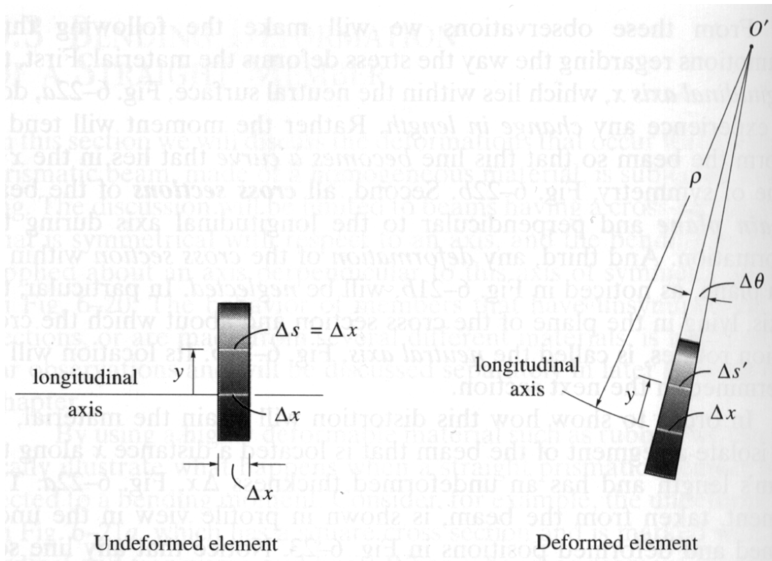

Figure 3.6 Undeformed and deformed elements

Note that ε is the in the strain in the x direction at distance y from the neutral axis and that $\varepsilon = \varepsilon_x$. Note that Eq. 3.3 is valid for elastic and inelastic action so long as the beam does not twist or buckle and the transverse shear stresses are relatively small.

An alternative method of developing Eq. 3.3 involves the definition of normal strain. An incremental element of a beam is shown both undeformed and deformed in Fig. 3.6. Note once again that any line segment Δx located on the neutral surface does not changes its length whereas any line segment Δs located at the arbitrary distance y from the neutral surface will elongate or contract and become $\Delta s'$ after deformation. Then by definition, the normal strain along Δs is determined as:

$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s} \tag{3.4}$$

Strain can be represented in terms of distance y from the neutral axis and radius of curvature ρ of the longitudinal axis of the element. Before deformation $\Delta s = \Delta x$ but after deformation Δx has radius of curvature ρ with center of curvature at point O'. Since $\Delta\theta$ defines the angle between the cross sectional sides of the incremental element, $\Delta s = \Delta x = \rho\,\Delta\theta$. Similarly, the deformed length of Δs becomes $\Delta s' = (\rho - y)\,\Delta\theta$. Substituting these relations into Eq. 3.4 gives:

$$\varepsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\,\Delta\theta}{\rho\,\Delta\theta} \tag{3.5}$$

Eq. 3.5 can be arithmetically simplified as $\varepsilon = -y/\rho$. Since the maximum strain occurs at the outer fibre which is distance c from the neutral surface, $\varepsilon_{max} = -c/\rho = \varepsilon_c$, the ratio of strain at y to maximum strain is

$$\frac{\varepsilon}{\varepsilon_{max}} = \frac{-y/\rho}{-c/\rho} \tag{3.6}$$

which when simplified and rearranged gives the same result as Eq. 3.3:

$$\varepsilon = \left(\frac{y}{c}\right)\varepsilon_{max} = \left(\frac{y}{c}\right)\varepsilon_c \tag{3.7}$$

Note that an important result of the strain equations for $\varepsilon = -y/\rho$ and $\varepsilon_{max} = -c/\rho = \varepsilon_c$ indicate that the longitudinal normal strain of any element within the beam depends on its location y on the cross section and the radius of curvature of the beam's longitudinal axis at that point. In addition, a contraction ($-\varepsilon$) will occur in fibres located "above" the neutral axis (+y) whereas elongation ($+\varepsilon$) will occur in fibres located "below" the neutral axis (-y).

Stress

The determination of stress distributions of beams in necessary for determining the level of performance for the component. In particular, stress-based failure theories require determination of the maximum combined stresses in which the complete stress state must be either measured or calculated.

Normal Stress: Having derived the proportionality relation for strain, ε_x, in the x-direction, the variation of stress, σ_x, in the x-direction can be found by substituting σ for ε in Eqs. 3.3 or 3.7. In the elastic range and for most materials uniaxial tensile and compressive stress-strain curves are identical. If there are differences in tension and compression stress-strain response, then stress must be computed from the strain distribution rather than by substitution of σ for ε in Eqs. 3.3 or 3.7.

Note that for a beam in pure bending since no load is applied in the z-direction, σ_z is zero throughout the beam. However, because of loads applied in the y-direction to obtain the bending moment, σ_y is not zero, but it is small enough compared to σ_x to neglect. In addition, σ_x while varying linearly in the y direction is uniformly distributed in the z-direction. Therefore, a beam under only a bending load will be in a uniaxial, albeit a non uniform, stress state.

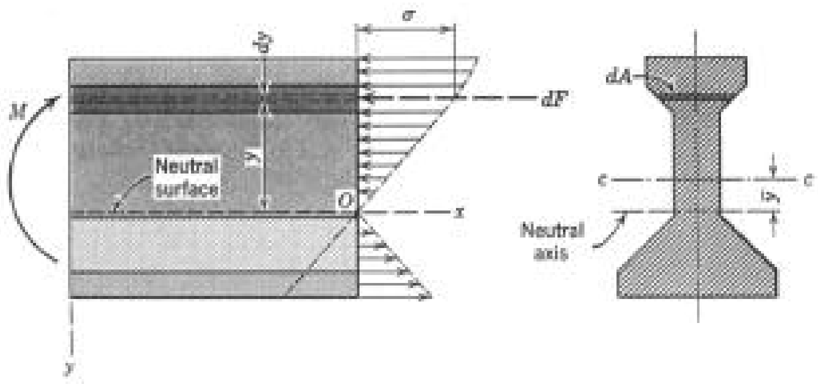

Figure 3.7 Stress (force) distribution in a bent beam

Note that for static equilibrium, the resisting moment, M_r, must equal the applied moment, M, such that $\sum M_O = 0$ where (see Fig. 3.7):

$$M_r = \int_A dFy = \int_A \sigma dAy \tag{3.8}$$

and since y is measured from the neutral surface, it is first necessary to locate this surface by means of the equilibrium equation $\sum F_x = 0$ which gives $\int_A \sigma dA = 0$. For the case of elastic action the relation between σ_x and y can be obtained from generalized Hooke's law $\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$ and the observation that $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$. The resulting stress-strain relation is for the uniaxial stress state such that $\sigma_x = E\varepsilon_x$ which when substituted into Eq. 3.3 or 3.7 gives

$$\sigma_x = E\left(\frac{\varepsilon_c}{c}\right)y = \frac{\sigma_c}{c}y \tag{3.9}$$

Substituting Eq. 3.9 into Eq. 3.8 gives:

$$M_r = \int_A \sigma dAy = \frac{\sigma_c}{c}\int_A y^2 dA = \frac{\sigma_x}{y}\int_A y^2 dA \tag{3.10}$$

Note that the integral is the second moment of the cross sectional area, also known as the moment of inertia, I, such that

$$\mathrm{I} = \int_A y^2 dA \tag{3.11}$$

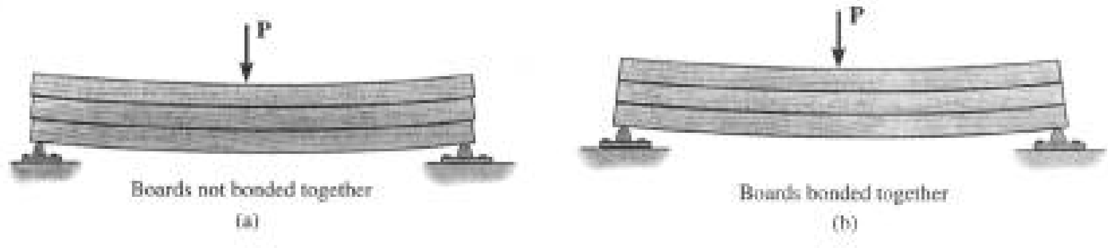

Figure 3.8 Action of shear stresses in unbonded and bonded boards

Substituting Eq. 3.11 into Eq. 3.10 and rearranging results in the elastic flexure stress equation:

$$\sigma_x = \frac{My}{I} \tag{3.12}$$

where σ_x is the normal bending stress at a distance y from the neutral surface and acting on a transverse plane and M is the resisting moment of the section. At any section of the beam, the fibre stress will be maximum at the surface farthest from the neutral axis such that.

$$\sigma_{max} = \frac{Mc}{I} = \frac{M}{Z} \tag{3.13}$$

where $Z=I/c$ is called the section modulus of the beam. Although the section modulus can be readily calculated for a given section, values of the modulus are often included in tables to simplify calculations.

Shear Stress: Although normal bending stresses appear to be of greatest concern for beams in bending, shear stresses do exist in beams when loads (i.e., transverse loads) other than pure bending moments are applied. These shear stresses are of particular concern when the longitudinal shear strength of materials is low compared to the longitudinal tensile or compressive strength (an example of this is in wooden beams with the grain running along the length of the beam). The effect of shear stresses can be visualized if one considers a beam being made up of flat boards stacked on top of one another without being fastened together and then loaded in a direction normal to the surface of the boards. The resulting deformation will appear somewhat like a deck of cards when it is bent (see Fig. 3.8a). The lack of such relative sliding and deformation in an actual solid beam suggests the presence of resisting shear stresses on longitudinal planes as if the boards in the example were bonded together as in Fib. 3.8b. The resulting deformation will distort the beam such that some of the assumptions made to develop the bending strain and stress relations (for example, plane sections remaining plane) are not valid as shown in Fig. 3.9.

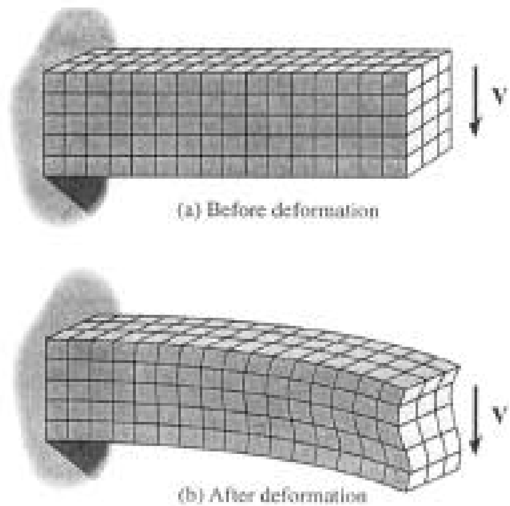

Figure 3.9 Distortion in a bend beam due to shear

The development of a general shear stress relation for beams is again based on static equilibrium such that $\sum F = 0$. Referring to the free body diagram shown in Fig. 3.10, the differential force, dF_1 is the normal force acting on a differential area dA and is equal to $\sigma \, dA$. The resultant of these differential forces is F_1 (not shown). Thus, $F_1 = \int \sigma \, dA$ integrated over the shaded area of the cross section, where σ is the fibre stress at a distance y from the neutral surface and is given by the expression $\sigma = \frac{My}{I}$.

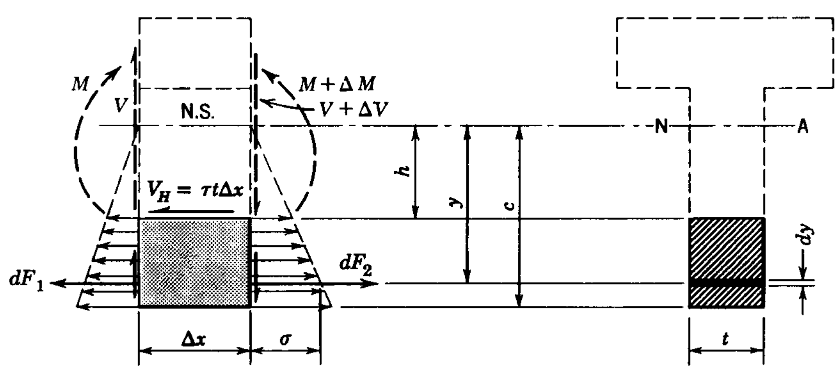

Figure 3.10 Free body diagram for development of shear stress relation

When the two expressions are combined, the force, F_1, becomes:

$$F_1 = \frac{M}{I}\int y\,dA = \frac{M}{I}\int_h^c ty\,dy \tag{3.14}$$

Similarly, the resultant force on the right side of the element is

$$F_2 = \frac{(M+\Delta M)}{I}\int_h^c ty\,dy \tag{3.15}$$

The summation of forces in the horizontal direction on Fig. 3.10 gives

$$V_H = F_2 - F_1 = \frac{\Delta M}{I}\int_h^c ty\,dy \tag{3.16}$$

The average shear stress is V_H divided by the area from which

$$\tau = \lim_{\Delta x \to 0}\frac{\Delta M}{\Delta x}\left(\frac{1}{It}\right)\int_h^c ty\,dy = \frac{dM}{dx}\left(\frac{1}{It}\right)\int_h^c ty\,dy \tag{3.17}$$

Recall that V=dM/dx, which is the shear at the beam section where the stress is being evaluated. Note that the integral, $Q=\int_h^c ty\,dy$ is the first moment of that portion of the cross sectional area between the transverse line where the stress is being evaluated and the extreme fiber of the beam. When Q and V are substituted into Eq. 3.17, the formula for the horizontal / longitudinal shear stress is:

$$\tau = \frac{VQ}{It} \tag{3.18}$$

Note that the flexure formula used in this derivation is subject to the same assumptions and limitations used to develop the flexure strain and stress relations. Also, although the stress given in Eq. 3.18 is associated with a particular point in a beam, it is averaged across the thickness, t, and hence it is accurate only if t is not too great. For uniform cross sections, such as a rectangle, the shear stress of Eq. 3.18 takes on a parabolic distribution, with $\tau = 0$ at the outer fibre (where y=c and $\sigma = \sigma_{max}$) and $\tau = \tau_{max}$ at the neutral surface (where y=0 and $\sigma = 0$) as shown in Fig. 3.11.

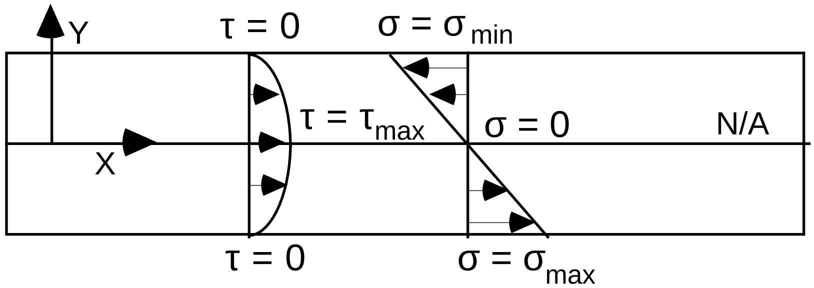

Figure 3.11 Shear and normal stress distributions in a uniform cross section beam

Finally, the maximum shear stress for certain uniform cross section geometries can be calculated and tabulated as shown in Fig. 3.12. Note that a first order approximation for maximum shear stress might be made by dividing the shear force by the cross sectional area of the beam to give an average shear stress such that $\tau_{av} \approx \frac{V}{A}$. However, if the maximum shear stress is interpreted as the critical shear stress, than an error of 50% would result for a beam with a rectangular cross section where $\tau_{max} \approx \frac{3V}{2A}$ which is 1.5 times $\tau_{av} \approx \frac{V}{A}$.

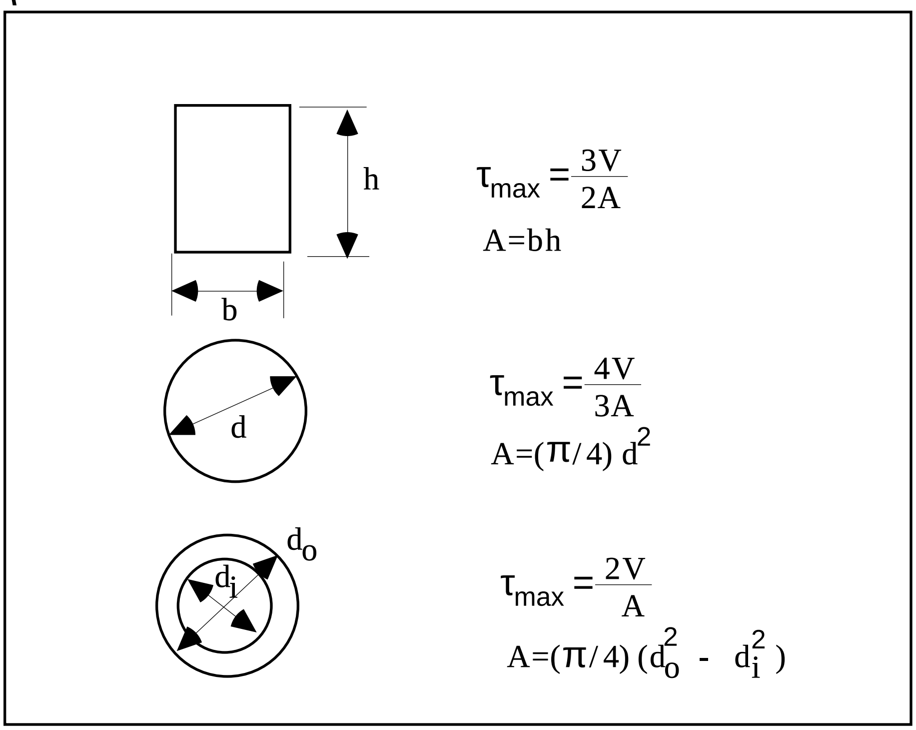

Figure 3.12 Maximum shear stresses for some common uniform cross sections

Deflections

Often limits must be placed on the amount of deflection a beam or shaft may undergo when it is subjected to a load. For example beams in many machines must deflect just the right amount for gears or other parts to make proper contact. Deflections of beams depend on the stiffness of the material and the dimensions of the beams as well as the more obvious applied loads and supports. In order of decreasing usage four common methods of calculating beam deflections are: 1) double integration method, 2) superposition method, 3) energy (e.g., unit load) method, and 4) area-moment method. The double integration method will be discussed in some detail here.

Deflections Due to Moments: When a straight beam is loaded and the action is elastic, the longitudinal centroidal axis of the beam becomes a curve defined as "elastic curve." In regions of constant bending moment, the elastic curve is an arc of a circle of radius, ρ, as shown in Fig. 3.13 in which the portion AB of a beam is bent only with bending moments. Therefore, the plane sections A and B remain plane and the deformation (elongation and compression) of the fibres is proportional to the distance from the neutral surface, which is unchanged in length. From Fig. 3.13:

$$\theta = \frac{L}{\rho} = \frac{L+\delta}{\rho + c} \tag{3.19}$$

from which

$$\frac{c}{\rho} = \frac{\delta}{L} = \varepsilon = \frac{\sigma}{E} = \frac{Mc}{EI} \tag{3.20}$$

and finally

$$\frac{1}{\rho} = \frac{M}{EI} \tag{3.21}$$

which relates the radius of curvature of the neutral surface of the beam to the bending moment, M, the stiffness of the material, E, and the moment of inertia of the cross section, I.

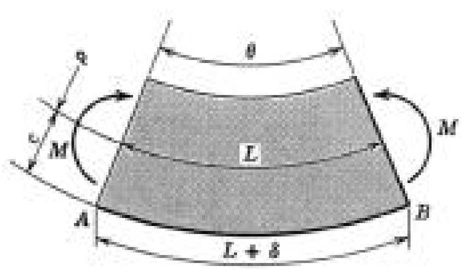

Figure 3.13 Bent element from which relation for elastic curve is obtained

Equation 3.21 is useful only when the bending moment is constant for the interval of the beam involved. For most beams the bending moment is a function of the position along the beam and a more general expression is required.

The curvature equation from calculus is

$$\frac{1}{\rho} = \frac{d^2 y / dx^2}{\left[1 + (dy/dx)^2\right]^{3/2}} \tag{3.22}$$

which for actual beams can be simplified because the slope dy/dx is small and its square is even smaller and can be neglected as a higher order term. Thus, with these simplifications, Eq. 3.22 becomes

$$\frac{1}{\rho} = \frac{d^2 y}{dx^2} = y'' \tag{3.23}$$

Substituting Eq. 3.23 into Eq. 3.21 and rearranging gives

$$EI\frac{d^2 y}{dx^2} = M_x = EIy'' \tag{3.24}$$

which is the differential equation for the elastic curve of a beam.

An alternative method for obtaining Eq. 3.24 is to use the geometry of the bent beam as shown in Fig. 3.14 where it is evident that dy/dx = tan $\theta \approx \theta$ for small angles and that $d^2 y / dx^2 = d\theta / dx$. From Fig. 3.14 it can be shown that

$$d\theta = \frac{dL}{\rho} = \frac{dx}{\rho} \tag{3.25}$$

for small angles and therefore.

$$y'' = \frac{d^2 y}{dx^2} = \frac{d\theta}{dx} = \frac{1}{\rho} = \frac{M_x}{EI} \qquad EI\frac{d^2 y}{dx^2} = EIy'' = M_x \tag{3.26}$$

For the coordinate system shown in Fig. 3.15, the signs of the moment and second derivative are as shown. It is also important to note the following physical quantities and beam action.

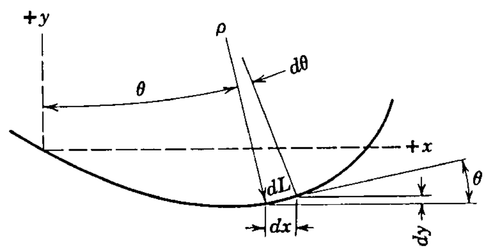

Figure 3.14 Bent beam from which relation for elastic curve is obtained.

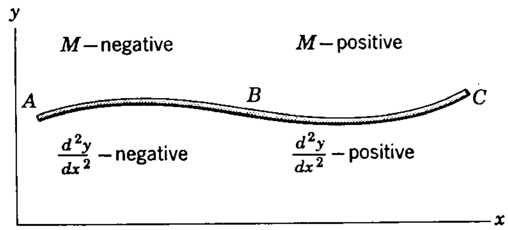

Figure 3.15 Sign conventions used for deflection

$$\text{deflection} = y$$

$$\text{slope} = \frac{dy}{dx} = y'$$

$$\text{moment} = M_x = EI\frac{d^2y}{dx^2} = EIy''$$

$$\text{shear} = \frac{dM}{dx} = EI\frac{d^3y}{dx^3} = EIy''' \text{ (for constant EI)} \tag{3.27}$$

$$\text{load} = \frac{dV}{dx} = EI\frac{d^4y}{dx^4} = EIy^{iv} \text{(for constant EI)}$$

It is interesting to note that from Eqs. 3.24 and 3.26 can be written as

$$M_x = EI\frac{d\theta}{dx} \tag{3.28}$$

from which

$$\int_{\theta_A}^{\theta_B} d\theta = \int_{x_A}^{x_B} \frac{M_x}{EI} dx \tag{3.29}$$

Eqs. 3.28 and 3.29 show that except for the factor EI, the area under the moment diagram between any two points along the beam gives the change in slope between the same two points. Likewise, the area under the slope diagram between two points along a beam gives the change in deflection between these points. These relations have been used to construct the series of diagrams shown in Fig. 3.16 for a simply supported beam with a concentrated load at the center of the span. The geometry of the beam was used to locate the points of zero slope and deflection, required as the starting points for the construction.

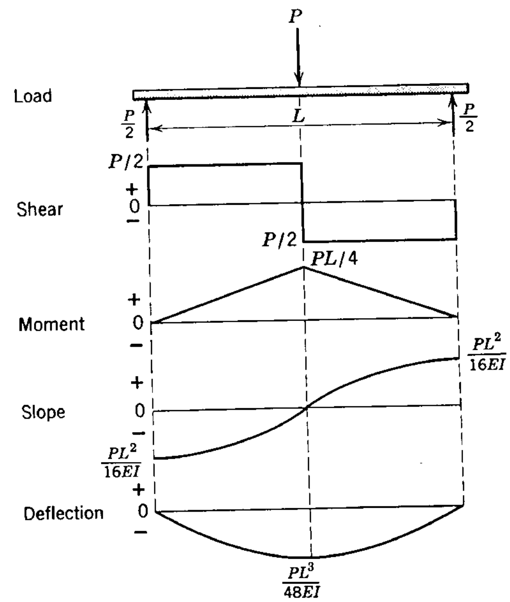

Figure 3.16 Illustration of various elastic relations for a beam in three-point loading

It is important to remember that the calculation of deflections from elastic curve relations is based on the following assumptions:

1) The square of the slope of the beam is assumed to be negligible compared to unity
2) The beam deflection due to shear stresses is negligible (i.e., plane sections remain plane)
3) The value of E and I remain constant for any interval along the beam.

The double integration method can be used to solve Eq. 3.24 for the deflection y as a function of distance along the beam, x. The constants of integration are evaluated by applying the applicable boundary conditions.

Boundary conditions are defined by a known set of values of x and y or x and dy/dx at a specific point in the beam. One boundary condition can be used to determine one and only one constant of integration. A roller or pin at any point in a beam (see Figs. 3.17a and 3.17b) represents a simple support which cannot deflect ($y=0$) but can rotate ($dy/dx \neq 0$). At a fixed end (see Figs. 3.17c and 3.17d) the beam can neither deflect or rotate ($y=0$ and $dy/dx=0$).

Matching conditions are defined as the equality of slope or deflection, as determined at the junction of two intervals from the elastic curve equations for both intervals.

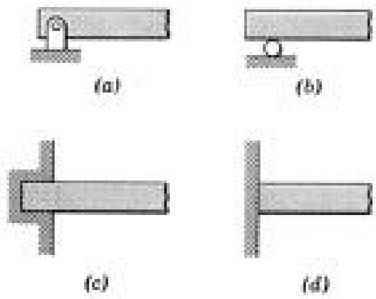

Figure 3.17 Types of boundary conditions

Calculating deflection of a beam by the double integration method involves four definite steps and the following sequence for these steps is recommended.

1) Select the interval or intervals of the beam to be used; next, place a set of coordinate axes on the beam with the origin at one end of an interval and then indicate the range of values of x in each interval. For example, two adjacent intervals might be: $0 \le x \le L$ and $L \le x \le 3L$
2) List the available boundary conditions and matching conditions (where two or more adjacent intervals are used) for each interval selected. Remember that two conditions are required to evaluate the two constants of integration for each interval used.
3) Express the bending moment as a function of x for each interval selected, and equate it to EI dy^2/dx^2 =EIy''.
4) Solve the differential equation or equations form item 3 and evaluate all constants of integration. Check the resulting equations for dimensional homogeneity. Calculate the deflection a specific points where required.

Deflections due to Shear: Generally deflections due to shear can be neglected as small (<1%) compared to deflections due to moments. However, for short, heavily-loaded beams, this deflection can be significant and an approximate method can be used to evaluate it. The deflection of the neutral surface, dy due to shearing stresses in the interval dx along the beam in Fig. 3.18 is

$$dy = \gamma dx = \frac{\tau}{G} dx = \frac{VQ}{GIt} dx \qquad (3.30)$$

from which the shear in Fig. 3.18 is negative such that

$$\frac{GIt}{Q}\frac{dy}{dx} = -V \qquad \frac{GIt}{Q} y' = -V \qquad (3.31)$$

Since the vertical shearing stress varies from top to bottom of a beam the deflection due to shear is not uniform. This non uniform distribution is reflected as slight warping of a beam. Equation 3.31 gives values too high because the maximum shear stress (at the neutral surface) is used and also because the rotation of the differential shear element is ignored. Thus, an empirical relation is often used in which a shape factor, k, is employed to account for this change of shear stress across the cross section such that

$$kAGy' = -V \qquad y' = \frac{-V}{kAG} \tag{3.32}$$

Often k is approximated as k≈1 but for box-like sections or webbed sections it is estimated as:

$$\frac{1}{\mathrm{k}} = \frac{\mathrm{A}_{\mathrm{total}}}{A_{web}} \tag{3.33}$$

A single integration method can be used to solve Eq. 3.32 for the deflection due to shear. The constants of integration are then determined by employing the appropriate boundary and matching conditions. The resulting equation provides a relation for the deflection due to shear as a function of the distance x along the length of the beam. Note however that unless the beam is very short or heavily loaded the deflection due to shear is generally only about 1% of the total beam deflection.

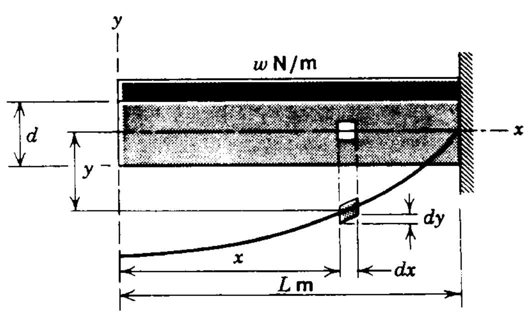

Figure 3.18 Deflection due to shear stress

An example of the use of integration methods is as follows for a simply supported beam in three-point loading. The loading condition, free body, shear and moment diagrams are shown in Fig. 3.19.

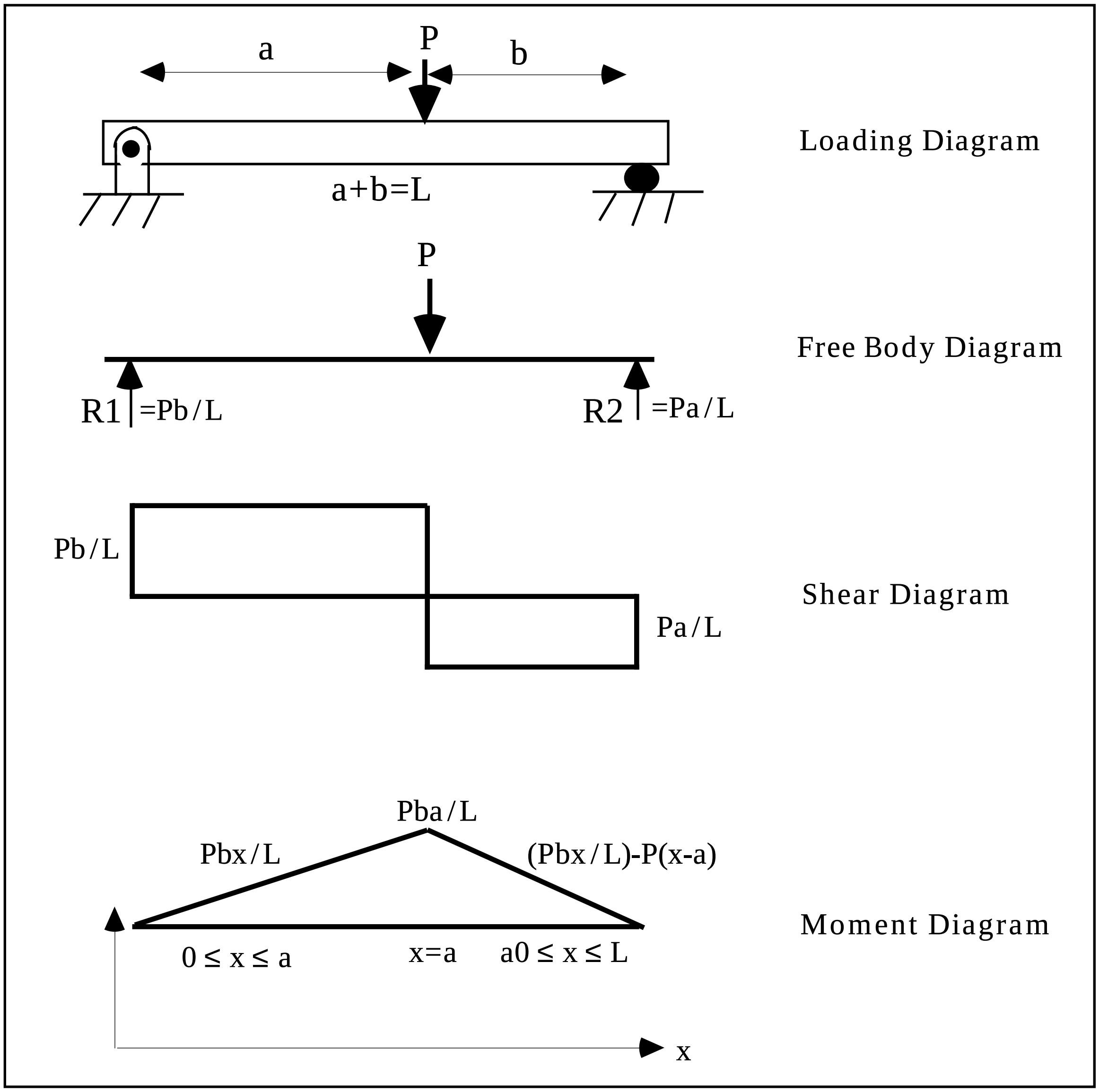

Figure 3.19 Loading condition, free body, shear and moment diagrams

There are two boundary conditions: at x=0, y_1=0 and at x=L, y_2=0

There are two matching conditions: at x=a, $y'_1=y'_2$ and at x=a, $y_1=y_2$

For $0 \le x \le a$ (region 1)

$V = Pb/L$
$M = Pbx/L$

Double Integration Method

$$EIy_1'' = -M = -\frac{Pbx}{L}$$

$$\int EIy_1'' = \int \frac{-Pbx}{L} \qquad EIy_1' = \frac{-Pbx^2}{2L} + C_1$$

$$\int EIy_1' = \int \frac{-Pbx^2}{2L} + C_1$$

$$EIy_1 = \frac{-Pbx^3}{6L} + C_1 x + C_3$$

For $a \le x \le L$ (region 2)

$V = Pa/L$
$M = (Pbx/L) - P(x-a)$

Double Integration Method

$$EIy_2'' = -M = -\frac{Pbx}{L} + P(x-a)$$

$$\int EIy_2'' = \int \frac{-Pbx}{L} + P(x-a)$$

$$EIy_2' = \frac{-Pbx^2}{2L} + \frac{P(x-a)^2}{2} + C_2$$

$$\int EIy_2' = \int \frac{-Pbx^2}{2L} + \frac{P(x-a)^2}{2} + C_2$$

$$EIy_2 = \frac{-Pbx^3}{6L} + \frac{P(x-a)^3}{6} + C_2 x + C_4$$

Applying the matching conditons at x=a, $y'_1 = y'_2$

$$y_1' = \frac{1}{EI}\left[\frac{-Pba^2}{2L} + C_1\right] = \frac{1}{EI}\left[\frac{-Pba^2}{2L} + \frac{P(a-a)^2}{2} + C_2\right] = y_2'$$

so that $C_1 = C_2$

and at x=a, $y_1 = y_2$

$$y_1 = \frac{1}{EI}\left[\frac{-Pba^3}{6L} + C_1 a + C_3\right] = \frac{1}{EI}\left[\frac{-Pba^3}{6L} + \frac{P(a-a)^3}{6} + C_2 a + C_4\right] = y_2$$

Since $C_1 = C_2$, then $C_3 = C_4$

Applying the boundary conditons at x=0, y_1=0

$$y_1 = 0 = \frac{1}{EI}\left[\frac{-Pb\,0^3}{6L} + C_1 0 + C_3\right]$$

$$\text{So } C_3 = 0$$

and at x=L, y_2=0

$$y_2 = 0 = \frac{1}{EI}\left[\frac{-PbL^3}{6L} + \frac{P(L-a)^3}{6} + C_2 L + C_4\right] =$$

$$\text{Since } (L-a) = b \text{ and } C_1 = C_2, \text{and } C_3 = C_4 = 0$$

$$\text{then } C_2 = \left[\frac{PbL}{6} - \frac{Pb^3}{6L}\right] = \frac{Pb}{6L}\left[L^2 - b^2\right]$$

Finally, the equations for deflection due to the bending moment are:

For $0 \le x \le a$ (region 1)

$$y_1 = \frac{-Pbx}{6EIL}\left[L^2 - b^2 - x^2\right]$$

For $a \le x \le L$ (region 2)

$$y_2 = \frac{-Pbx}{6EIL}\left[L^2 - b^2 - x^2\right] - \frac{P(x-a)^3}{6EI}$$

The deflection due to the shear component is:

For $0 \le x \le a$ (region 1)

V=Pb/L

$$kAG\,y'_1 = -V \qquad y'_1 = \frac{-V}{kAG} = -\frac{Pb}{L}\frac{1}{kAG}$$

$$\int y'_1 = \int \frac{-Pb}{L}\frac{1}{kAG} \qquad y_1 = \frac{-Pb}{kLAG}x + C_1$$

For $a \le x \le L$ (region 2)

V=Pa/L

$$kAG\,y'_1 = -V \qquad y'_1 = \frac{-V}{kAG} = -\frac{Pa}{L}\frac{1}{kAG}$$

$$\int y'_1 = \int \frac{-Pa}{L}\frac{1}{kAG} \qquad y_1 = \frac{-Pa}{kLAG}x + C_2$$

Applying the boundary condition at x=0, y_1=0,

$$y_1 = 0 = \frac{-Pb}{\mathrm{kLAG}} 0 + C_1$$

$$\text{So } C_1 = 0 \text{ and } y_1 = \frac{-\mathrm{Pb}}{\mathrm{kLAG}} x \text{ for } 0 \le x \le a$$

Applying the boundary condition at x=L, y_2=0,

$$y_2 = 0 = \frac{-Pa}{\mathrm{kLAG}} L + C_2$$

$$\text{So } C_2 = \frac{\mathrm{Pa}}{\mathrm{kAG}} \text{ and } y_2 = \frac{\mathrm{Pa}}{\mathrm{kAG}}\left(-\frac{x}{L} + 1\right)$$

Now the total deflection relation for both bending and shear is:

For $0 \le x \le a$ (region 1)

$$y_1 = \left[\frac{-\mathrm{Pbx}}{6\mathrm{EIL}}\left[L^2 - b^2 - x^2\right]\right] - \frac{Pb}{\mathrm{kLAG}} x$$

For $a \le x \le L$ (region 2)

V=Pa/L

$$y_2 = \left[\frac{-\mathrm{Pbx}}{6\mathrm{EIL}}\left[L^2 - b^2 - x^2\right] - \frac{P(x-a)^3}{6\mathrm{EI}}\right] - \frac{Pa}{\mathrm{kLAG}}\left[\frac{-x}{L} + 1\right]$$