Integration by Trigonometric Substitution

In this section, we see how to integrate expressions like

$$\boxed{\sqrt{a^2-u^2}},\ \boxed{\sqrt{a^2+u^2}},\ \text{and}\ \boxed{\sqrt{u^2-a^2}}$$

depending on the function we need to integrate, we substitute one of the following trigonometric expressions to simplify the integration:

- For $\boxed{\sqrt{a^2-u^2}}$, use $\boxed{u=a\sin\theta}$
- For $\boxed{\sqrt{a^2+u^2}}$, use $\boxed{u=a\tan\theta}$
- For $\boxed{\sqrt{u^2-a^2}}$, use $\boxed{u=a\sec\theta}$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{9-x^2}}$

$$\because \sqrt{9-x^2} \equiv \sqrt{a^2-u^2} \Longrightarrow \text{ we use } \boxed{u=a\sin\theta}$$

$$\because a=3 \text{ and } u=x \Longrightarrow \boxed{x=3\sin\theta}$$

$$\Longrightarrow \boxed{\theta=\sin^{-1}(\frac{x}{3})} \text{ and } \boxed{dx=3\cos\theta d\theta}$$

$$\therefore \sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta}$$

$$= \sqrt{9(1-\sin^2\theta)} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = \boxed{3\cos\theta}$$

$$\therefore \int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta d\theta}{3\cos\theta} = \int d\theta = \theta + C = \boxed{\sin^{-1}(\frac{x}{3}) + C}$$

2) $\int \frac{dx}{\sqrt{25+x^2}}$

$\because \sqrt{25+x^2} \equiv \sqrt{a^2+u^2} \implies$ we use $\boxed{u = a\tan\theta}$

$\because a = 5$ and $u = x \implies \boxed{x = 5\tan\theta}$

$\implies \boxed{\theta = \tan^{-1}(\frac{x}{5})}$ and $\boxed{dx = 5\sec\theta d\theta}$

$\therefore \sqrt{25+x^2} = \sqrt{25+(5\tan\theta)^2} = \sqrt{25+25\tan^2\theta}$

$= \sqrt{25(1+\tan^2\theta)} = 5\sqrt{1+\tan^2\theta} = 5\sqrt{\sec^2\theta} = \boxed{5\sec\theta}$

$\therefore \int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5\sec^2\theta d\theta}{5\sec\theta} = \int \sec\theta d\theta$

$= \ln|\sec\theta + \tan\theta| + C = \boxed{\ln|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}| + C}$

3) $\int \frac{dx}{x\sqrt{x^2-16}}$

$\because \sqrt{x^2-16} \equiv \sqrt{u^2-a^2} \implies$ we use $\boxed{u = a\sec\theta}$

$\because a = 4$ and $u = x \implies \boxed{x = 4\sec\theta}$

$\implies \boxed{\theta = \sec^{-1}(\frac{x}{4})}$ and $\boxed{dx = 4\sec\theta\tan\theta d\theta}$

$\therefore \sqrt{x^2-16} = \sqrt{(4\sec\theta)^2-16} = \sqrt{16\sec^2\theta-16}$

$= \sqrt{16(\sec^2\theta-1)} = 4\sqrt{\sec^2\theta-1} = 4\sqrt{\tan^2\theta} = \boxed{4\tan\theta}$

$\therefore \int \frac{dx}{x\sqrt{x^2-16}} = \int \frac{4\sec\theta\tan\theta d\theta}{4\sec\theta * 4\tan\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C = \boxed{\frac{1}{4}\sec^{-1}(\frac{x}{4}) + C}$

Integration of Rational Functions

Definition: A rational function is a quotient of two polynomials, written as $R(X)=\frac{Pn(x)}{Qm(x)}$, $Qm(x)\neq 0$ where $Pn(x)$ and $Qm(x)$ are polynomials Of degree n and m respectively.

(1) If $n \geq m$, we perform a long division until we obtain a rational function whose numerator degree less than to the denominator

Example (1): Evaluate the integral $\int \frac{(x^2-3x+5)}{(x-2)}\, dx$

Solution: $\int \frac{(x^2-3x+5)}{(x-2)}\, dx = \int \left[(x-1)+\frac{3}{x-2}\right] dx$

$= \frac{1}{2}x^2 - x + 3Ln|x-2| + C$

$x - 1$

x-2 | $x^2 - 3x + 5$

$-x^2 + 2x$

x+5

$+x + 2$

3

Example (2): Evaluate the integral $\int \frac{x^2+2}{x^2+1}\, dx$

Solution: $\int \frac{x^2+2}{x^2+1}\, dx = \int \left(1+\frac{1}{x^2+1}\right) dx$

$= x + tan^{-1}(x) + C$

1

$x^2 + 1$ | $x^2 + 2$

$\mp x^2 \mp 1$

1

(2) If $n < m$, we shall discuss the three cases of separating $\frac{Pn(x)}{Qm(x)}$ into a sum of partial fractions.

Case (1): If the m factor of $Qm(x)$ are all different and simple, that is , $Q_m(x) = (x-a_1)\,(x-a_2)\,\ldots\,(x-a_m)$. Then we assign the sum of m partial fractions to these factors as follows $\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_m}{(x-a_m)}$. where $A_1, A_2, \ldots, A_m$ are constants must be evaluated.

Example (2): Evaluate the integral $\int \frac{x^2+3x+3}{x^3-x}\, dx$

Solution: $\frac{x^2+3x+3}{x^3-x} = \frac{x^2+3x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$$= \frac{A(x-1)(x+1)+B(x)(x+1)+C(x)(x-1)}{x(x-1)(x+1)}$$

$$x^2 + 3x + 3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$
$$= (A + B + C)x^2 + (B - C)x - A$$

$$\left.\begin{aligned} (A + B + C) &= 1 \\ B - C &= 3 \\ -A &= 3 \end{aligned}\right\} \quad A = -3\,,\ B = \frac{7}{2},\quad C = \frac{1}{2}$$

$$\therefore \int \frac{x^2 + 3x + 3}{x^3 - x}\, dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right) dx$$

$$= \int \left(\frac{-3}{x} + \frac{\frac{7}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\right) dx$$

$$= -3\ln x + \frac{7}{2} ln(x-1) + \frac{1}{2} ln(x+1) + C$$

Case (2): Repeated factors of $Q_m(x)$ suppose $(x-a)^r$ is the highest power of $(x-a)$which divides $Q_m(x)$.

Then we assign the sum of r partial fractions to these factors as follows:

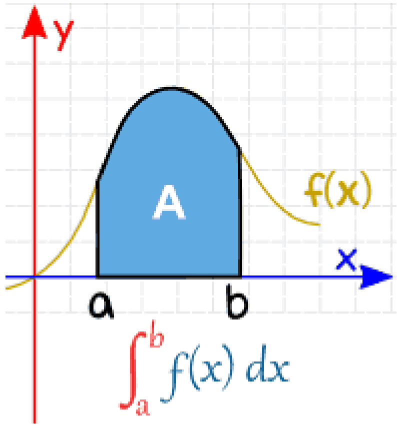
$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$ where $A_1, A_2, \ldots, A_r$ are constants must be evaluated.

For example $\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+2)^3} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)} + \frac{E}{(x+2)} + \frac{F}{(x+2)^2} + \frac{G}{(x+2)^3}$

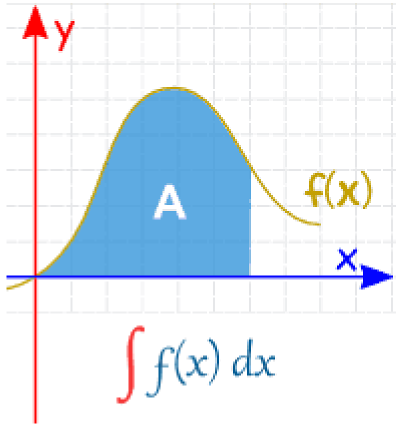
The Definite Integrals and Its Application

A **Definite Integral** has start and end values: in other words, there is an **interval** [a, b].

a and b (called limits, bounds or boundaries) are put at the bottom and top of the "S", like this:



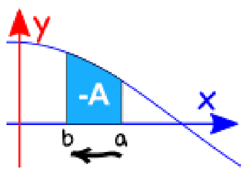
Definite Integral
(from **a** to **b**)



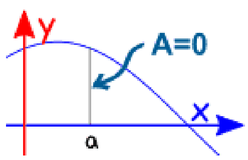
Indefinite Integral
(no specific values)

Properties

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

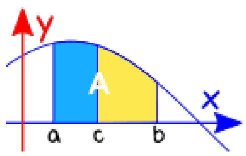


$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Example (1): Evaluate the integral $\int_{-3}^{2}(6-x-x^2)\,dx$

Solution: $\int_{-3}^{2}(6-x-x^2)dx=\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$

$=\left[6(2)-\frac{(2)^2}{2}-\frac{(2)^3}{3}\right]-\left[6(-3)-\frac{(-3)^2}{2}-\frac{(-3)^3}{3}\right]$

$=\left[12-2-\frac{8}{3}\right]-\left[-18-\frac{9}{2}+9\right]=\left[10-\frac{8}{3}\right]-\left[-9-\frac{9}{2}\right]$

$=19-\frac{8}{3}+\frac{9}{2}=19+\frac{-16+27}{6}=19+\frac{11}{6}=\frac{125}{6}$.

Example (2): Evaluate the integral $\int_{0}^{\pi}\sin x\,dx$

Solution: $\int_{0}^{\pi}\sin x\,dx=-\cos x|_{0}^{\pi}=-(\cos\pi-\cos o)$

$=-(-1-1)=-(-2)=2$

Example (3): Evaluate the integral $\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{cos^2x}dx$

Solution: $\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{cos^2 2x}dx=\int_{0}^{\frac{\pi}{6}}(\cos 2x)^{-2}\sin 2x\;dx$

$=\left(\frac{1}{-2}\right)\int_{0}^{\frac{\pi}{6}}(\cos 2x)^{-2}\;(-2)\sin 2x\;dx$

$=\frac{-1}{2}\left(\frac{(\cos 2x)^{-1}}{-1}\right)_{0}^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{1}{\cos 2x}\right)_{0}^{\frac{\pi}{6}}$

$=\frac{1}{2}\left[\frac{1}{\cos\left(2\frac{\pi}{6}\right)}-\frac{1}{\cos(2(0))}\right]=\frac{1}{2}\left[\frac{1}{\cos\left(\frac{\pi}{3}\right)}-\frac{1}{\cos(0)}\right]=\frac{1}{2}\left[\frac{1}{\frac{1}{2}}-\frac{1}{1}\right]$

$=\frac{1}{2}[2-1]=\frac{1}{2}$