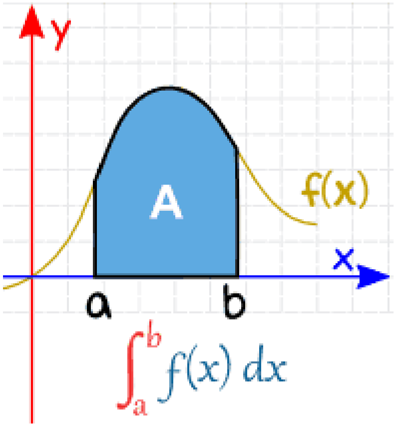
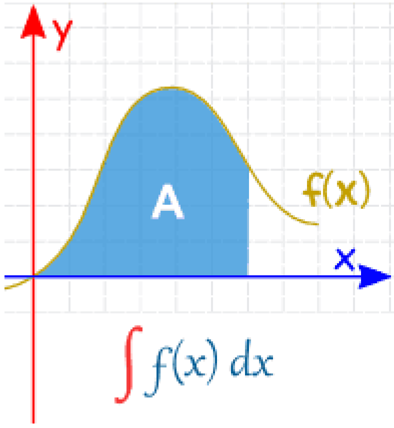
The Definite Integrals and Its Application

A **Definite Integral** has start and end values: in other words, there is an **interval** [a, b].

a and b (called limits, bounds or boundaries) are put at the bottom and top of the "S", like this:



Definite Integral
(from **a** to **b**)

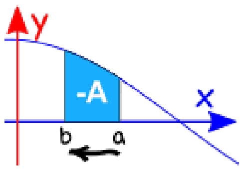


Indefinite Integral
(no specific values)

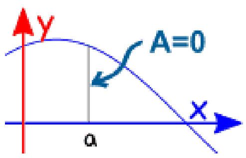
Properties

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

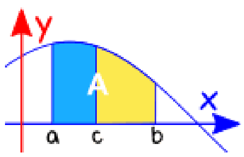
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



$$\int_a^a f(x)\,dx = 0$$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Example (1): Evaluate the integral $\int_{-3}^{2}(6-x-x^2)\,dx$

Solution: $\int_{-3}^{2}(6-x-x^2)dx=\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$

$= \left[6(2)-\frac{(2)^2}{2}-\frac{(2)^3}{3}\right]-\left[6(-3)-\frac{(-3)^2}{2}-\frac{(-3)^3}{3}\right]$

$= \left[12-2-\frac{8}{3}\right]-\left[-18-\frac{9}{2}+9\right]=\left[10-\frac{8}{3}\right]-\left[-9-\frac{9}{2}\right]$

$= 19-\frac{8}{3}+\frac{9}{2}=19+\frac{-16+27}{6}=19+\frac{11}{6}=\frac{125}{6}$.

Example (2): Evaluate the integral $\int_{0}^{\pi}\sin x\,dx$

Solution: $\int_{0}^{\pi}\sin x\,dx = -\cos x|_0^{\pi} = -(\cos\pi - \cos o)$

$= -(-1-1)= -(-2)=2$

Example (3): Evaluate the integral $\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{cos^2 x}dx$

Solution: $\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{cos^2 2x}dx = \int_{0}^{\frac{\pi}{6}}(\cos 2x)^{-2}\sin 2x\,dx$

$=\left(\frac{1}{-2}\right)\int_{0}^{\frac{\pi}{6}}(\cos 2x)^{-2}\,(-2)\sin 2x\,dx$

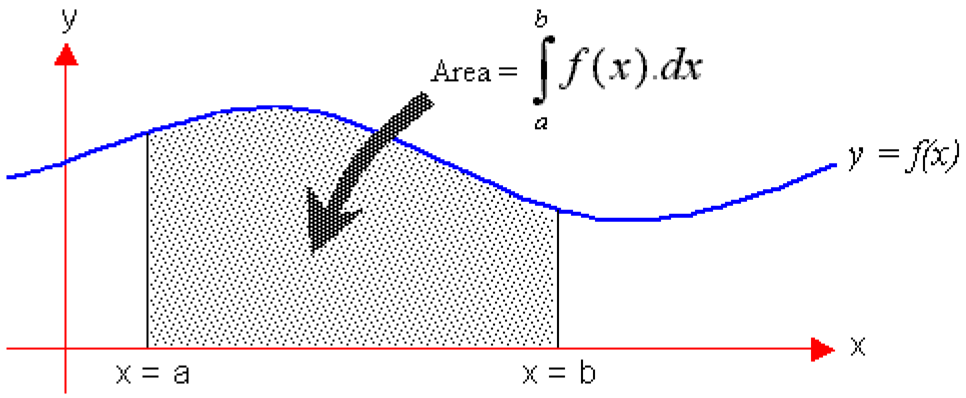
$=\frac{-1}{2}\left(\frac{(\cos 2x)^{-1}}{-1}\right)_0^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{1}{\cos 2x}\right)_0^{\frac{\pi}{6}}$

$=\frac{1}{2}\left[\frac{1}{\cos\left(2\frac{\pi}{6}\right)}-\frac{1}{\cos(2\,(0))}\right]=\frac{1}{2}\left[\frac{1}{\cos\left(\frac{\pi}{3}\right)}-\frac{1}{\cos(0)}\right]=\frac{1}{2}\left[\frac{1}{\frac{1}{2}}-\frac{1}{1}\right]$

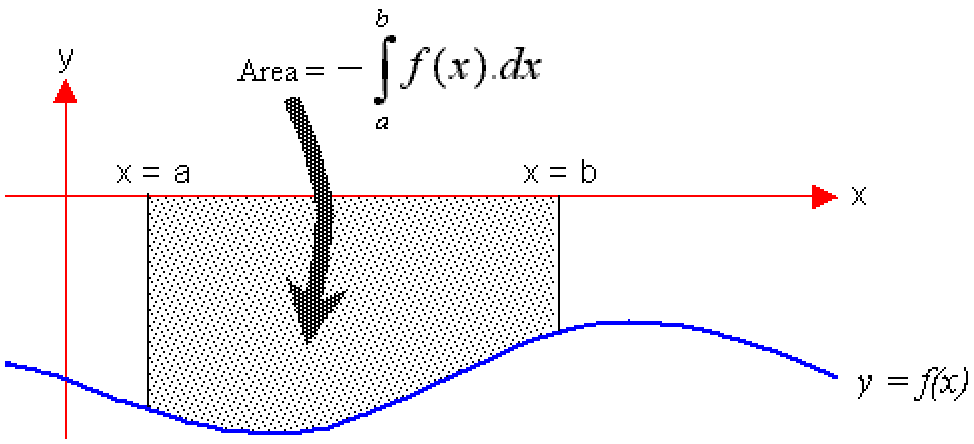
$=\frac{1}{2}\,[2-1]=\frac{1}{2}$

Area Under a Curve

The area under a curve between two points can be found by doing a definite integral between the two points. To find the area under the curve y = f(x) between x = a and x = b, integrate y = f(x) between the limits of a and b.



Remark: If the area is above x-axis, then the area is positive, and if the area under the x-axis, the area is negative, so we should change the sign to positive value by adding a negative sign or by taking the absolute value.



Remark: To avoid the negative value, we will take the absolute value:

$$\text{Area} = \left| \int_{x=a}^{x=b} f(x)dx \right|$$

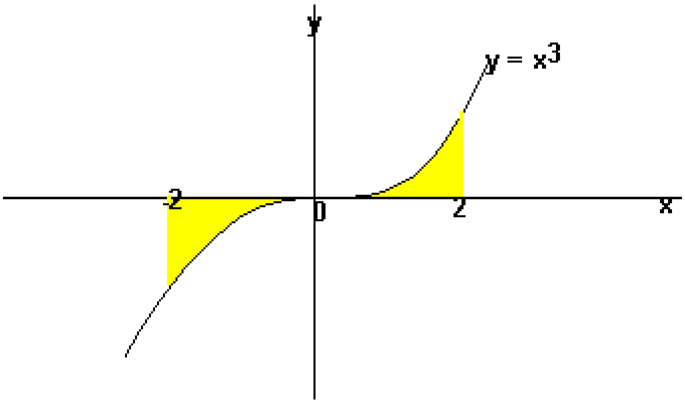
Example: Find the area bounded by $y = x^2$ and $x = 1$ and $x = 3$

Solution:

$$\text{Area} = \left| \int_{x=1}^{x=3} x^2 dx \right|$$
$$= \left| \left[\frac{x^3}{3} \right]_{x=1}^{x=3} \right|$$
$$= \left| \frac{3^3}{3} - \frac{1^3}{3} \right|$$
$$= \left| 9 - \frac{1}{3} \right| = \boxed{8\frac{2}{3}}\ unit^2$$

Example: Find the total area between the curve $y = x^3$ and $x = -2$ and $x = 2$

Solution:



If we simply integrated $y = x^3$ between $x = -2$ and $x = 2$, we would get:

$$\text{Area} = \left| \int_{x=-2}^{x=2} x^3 dx \right| = \left| \left[\frac{x^4}{4} \right]_{x=-2}^{x=2} \right| = \left| \frac{16}{4} - \frac{16}{4} \right| = 0$$

So, instead we have to split the graph up and do two separate integrals:

$$A1 = \left|\int_{x=0}^{x=2} x^3 dx\right| = \left|\left[\frac{x^4}{4}\right]_0^2\right| = \left|\frac{16}{4} - 0\right| = 4$$

$$A2 = \left|\int_{x=-2}^{x=0} x^3 dx\right| = \left|\left[\frac{x^4}{4}\right]_{-2}^0\right| = \left|0 - \frac{16}{4}\right| = \left|-4\right| = 4$$

$$\text{Hence, Area} = A1 + A2 = 4 + 4 = \boxed{8}\ unit^2$$

Example: Find the area bounded by the line x + y = 1 and the coordinate axes

Solution:

$$\because x + y = 2 \implies y = 2 - x$$

$$y = 0 \longrightarrow x = 2 \implies (2, 0)$$

$$\begin{aligned}\text{Area} &= \left|\int_{x=0}^{x=2} (2 - x) dx\right| \\ &= \left|\left(x - \frac{x^2}{2}\right)\right]_{x=0}^{x=2}\Bigg| \\ &= \left|(0 - 0) - \left(2 - \frac{2^2}{2}\right)\right| \\ &= \left|-2 + 2\right| = \boxed{4}\ unit^2\end{aligned}$$