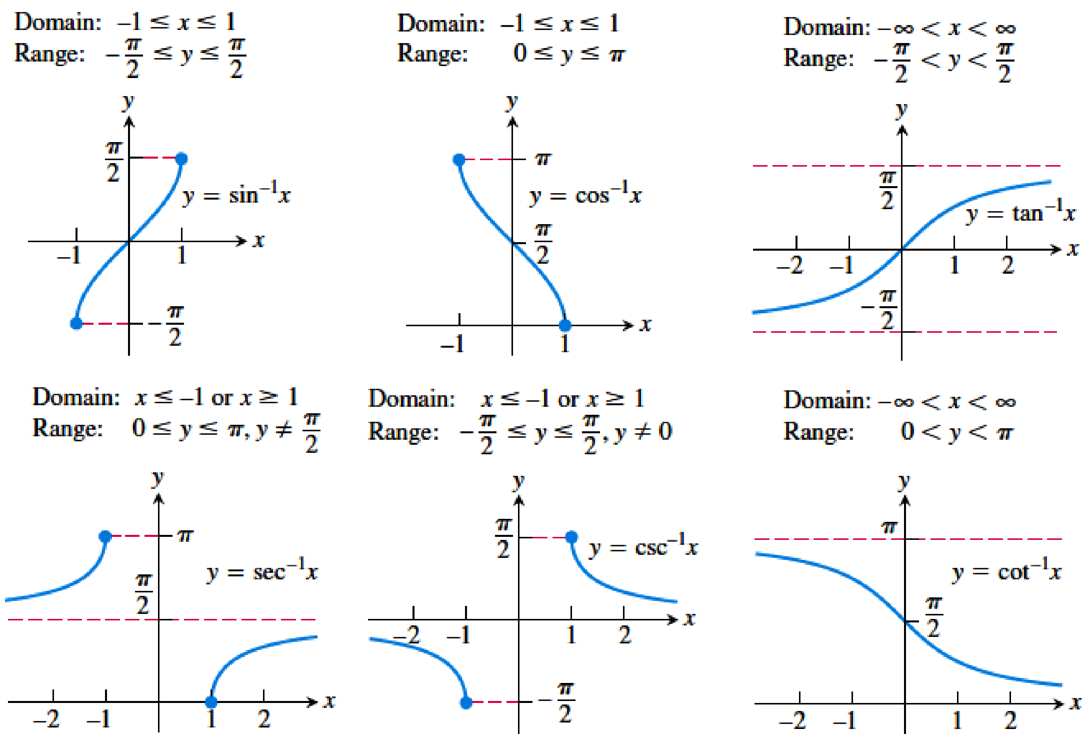
Inverse Trigonometric Functions

The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses.
The inverse sine function, denoted by $sin^{-1}x$ (or arcsin x), is defined to be the inverse of the restricted sine function. A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the 1" is not an exponent and so, $sin^{-1}x \neq \frac{1}{sinx}$

In inverse trigonometric functions the 1" looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine we would use the following. $(\sin x)^{-1} = \frac{1}{sinx}$



Derivatives of inverse trigonometric functions

Let u be a function x, the derivatives of inverse trigonometric functions are:

1. $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
2. $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
3. $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$
4. $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$
5. $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$
6. $\frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Example : Find the derivative for

1. $y = \sin^{-1} 2x \quad ⇨ \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{2}{\sqrt{1-4x^2}}$

2. $y = 3x\cos^{-1} 3x - \sqrt{1-9x^2}$

$$\frac{dy}{dx} = 3x \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3 + 3\cos^{-1} 3x - \frac{-18x}{2\sqrt{1-9x^2}}$$

$$= \frac{-9x}{\sqrt{1-9x^2}} + 3\cos^{-1} 3x + \frac{9x}{\sqrt{1-9x^2}} = 3\cos^{-1} 3x$$

Integrals of Inverse Trigonometric Functions

We can derive all the integration forms from our derivatives forms as follows

(1) $\frac{d}{du}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C}$

(2) $\frac{d}{du}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = -\cos^{-1}(u) + C}$

(3) $\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = \tan^{-1}(u) + C}$

(4) $\frac{d}{du}\cot^{-1}(u) = -\frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = -\cot^{-1}(u) + C}$

(5) $\frac{d}{du}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = \sec^{-1}(u) + C}$

(6) $\frac{d}{du}\csc^{-1}(u) = -\frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = -\csc^{-1}(u) + C}$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1-4x^2}}$

$= \frac{1}{2}\int \frac{2dx}{\sqrt{1-(2x)^2}}$

$= \boxed{\frac{1}{2}\sin^{-1}(2x)+C}$ or $\boxed{\frac{1}{2}\cos^{-1}(2x)+C}$

2) $\int \frac{dt}{1+t^2}$

$= \boxed{\tan^{-1}(t)+C}$ or $\boxed{-\cot^{-1}(t)+C}$

3) $\frac{dx}{x\sqrt{4x^2-1}}$

$= \frac{2dx}{2x\sqrt{(2x)^2-1}}$

$= \boxed{\sec^{-1}|2x|+C}$ or $\boxed{-\csc^{-1}|2x|+C}$

4) $\frac{-dx}{\sqrt{4-25x^2}}$

$= \frac{-dx}{\sqrt{4(1-\frac{25}{4}x^2)}}$

$= \frac{-dx}{2\sqrt{1-(\frac{5}{2}x)^2}}$

$= \frac{-1}{2}.\frac{2}{5}.\frac{\frac{5}{2}dx}{\sqrt{1-(\frac{5}{2}x)^2}}$

$= \boxed{-\frac{1}{5}\sin^{-1}(\frac{5}{2}x)+C}$ or $\boxed{\frac{1}{5}\cos^{-1}(\frac{5}{2}x)+C}$

5) $\int \frac{\cos(x)dx}{\sqrt{1-\sin^2(x)}}$

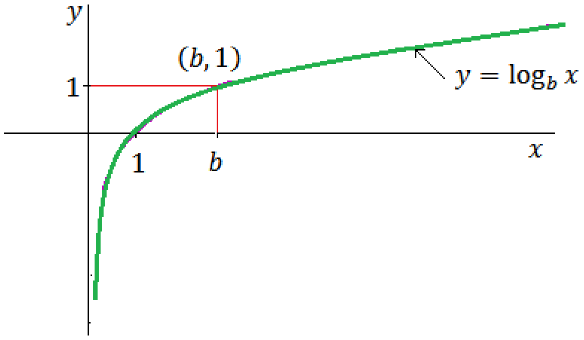
$= \sin^{-1}(\sin(x))+C$

$= \boxed{x+C}$

Logarithm function

The logarithm function with base b is the function $y=\log x$ where $b>0$ and $b\neq1$. The function is defined for all $x > 0$.

Here is its graph for any base b



Note the following:

1. For any base, the x-intercept is 1. ⇨ $\log_b 1 = 0$.
2. The graph passes through the point $(b, 1)$. ⇨ $\log_b b = 1$.
3. The graph is below the x-axis -- the logarithm is negative -- for $0 < x < 1$
4. The function is defined only for positive values of x.
5. The range of the function is all real numbers.
6. The negative y-axis is a vertical asymptote.
7. $\log_b(xy) = \log_b x + \log_b y$.
8. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.
9. $\log_b\left(\frac{1}{x}\right) = -\log_b x$.
10. $\log_b x^y = y\log_b x$.

The natural logarithm $y = \ln x$

The system of natural logarithms has the number called e as its base. e is an irrational number; its decimal value is approximately 2.71828182845904.
To indicate the natural logarithm of a number we write "ln." $\ln x$ means $\log x$. So we have

1. $\ln e = 1$
2. $\log_b x = \dfrac{\ln x}{\ln b}$
3. $\ln(xy) = \ln x + \ln y$
4. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$
5. $\ln x^n = n \ln x$

Derivative of natural logarithm function

If u is a function x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \ln(5x + 1)$

$$\frac{dy}{dx} = \frac{1}{5x+1} \times 5 = \frac{5}{5x+1}$$

2. $y = 2x \tan^{-1} x - \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1} x - \frac{2x}{x^2+1} = 2\tan^{-1} x$$

3. $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3\cos 3x = 3\cot 3x$$

Integrals of Logarithmic Functions

$$\because \frac{d}{du}\ln(u) = \frac{1}{u}\,du \xrightarrow{\int} \boxed{\int \frac{1}{u}du = \ln|u| + C},\ u \neq 0$$

Examples: Evaluate the following integrals:

1) $\int \frac{2}{x}dx$

$= 2\int \frac{1}{x}dx$

$= \boxed{2\ln|x| + C}$

2) $\int (\frac{3}{x^2} + \frac{5}{x})dx$

$= 3\int x^{-2}dx + 5\int \frac{1}{x}dx$

$= 3\frac{x^{-1}}{-1} + 5\ln|x| + C$

$= \boxed{\frac{-3}{x} + 5\ln|x| + C}$

3) $\int \frac{x}{(2x^2+3)}dx$

$= \frac{1}{4}\int \frac{4x}{(2x^2+3)}dx$

$= \boxed{\frac{1}{4}\ln|2x^2 + 3| + C}$

4) $\int \frac{\sec^2(x)}{\tan(x)}dx$

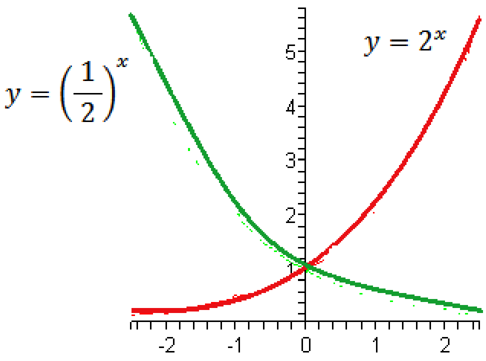
$= \boxed{\ln|\tan(x)| + C}$

Exponential functions

For any positive number $a>0$, $a\neq 1$, there is a function called an exponential function that is defined as $f(x)=a$

For example

$y = 2^x$,. $y = (\frac{1}{2})^x$



Basic rules for exponents

1. The product rule $a^x . a^y = a^{x+y}$

2. The quotient rule $\frac{a^x}{a^y} = a^{x-y}$

3. The rule for power of a power $(a^x)^y = a^{x.y}$

Natural exponential function

The function $f(x) = e^x$ is often called exponential function or natural exponential function which is an important function. The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

Derivatives of exponential function

If u is a function x, then

$$\frac{d}{dx}(e^u) = e^u.\frac{du}{dx}$$

Example : Find y' and y'' of the function

1. $y = e^{3x-2}$

$y' = e^{3x-2} \times 3 = 3e^{3x-2}$

$y'' = 3e^{3x-2} \times 3 = 9e^{3x-2}$

Integrals of Exponential Function

$$\because \frac{d}{du}e^{(u)} = e^{(u)}du \xrightarrow{\int} \int e^{(u)}du = e^{(u)} + C$$

Examples: Evaluate the following integrals:

1) $\int e^{2x}dx$

$= \frac{1}{2}\int 2e^{2x}dx$

$= \boxed{\frac{1}{2}e^{2x} + C}$

2) $\int e^{\sin(3x)}.\cos(3x)dx$

$= \frac{1}{3}\int 3.e^{\sin(3x)}.\cos(3x)dx$

$= \boxed{\frac{1}{3}e^{\sin(3x)} + C}$

3) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$= \boxed{\ln|e^x + e^{-x}| + C}$

4) $\int e^{2x}\sin(e^{2x})dx$

$= \frac{1}{2}\int 2.e^{2x}\sin(e^{2x})dx$

$= \boxed{-\cos(e^{2x}) + C}$