Techniques of Integration

I- Integration by parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated. For example, if the problem is to find

$$\int x^n \, cosx \, dx$$

We have two ways to do this:

1- Tabular integration by parts

Which is used for

1) the products of polynomials and sine function, $x \, sinbx$.
2) the products of polynomials and cosine function, $x \, cosbx$.
3) the products of polynomials and exponential function, $x \, e^{ax}$.

In any of these three cases we choose the polynomial as u and the product of sine function and dx (cosine or exponential function and dx respectively) as dv.

Example1: Evaluate

1-$\int x^3 \, cosxdx$

$1. \int x^3 \cos x \, dx$	Derivatives of u		Integrals of v
	x^3	$+$	$\cos x$
	$3x^2$	$-$	$\sin x$
	$6x$	$+$	$-\cos x$
	6	$-$	$-\sin x$
	0		$\cos x$

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$$

2- $\int x^2\, e^{2x}\, dx$

$D.of\ x^2$		$I.of\ e^{2x}$
x^2	+	e^{2x}
$2x$	−	$\frac{1}{2}e^{2x}$
2	+	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

$$\int x^2 e^{2x}\, dx = \frac{x^2}{2}e^{2x} - \frac{2x}{4}e^{2x} + \frac{2}{8}e^{2x} + c = \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + c$$

2-Integration by using the formula

$$\int u\, dv = u\, v - \int v\, du$$

Which is used for inverse trigonometric functions and logarithm functions.

Example2: Evaluate

1. $\int lnx\, dx$

$u = \ln x$ and $dv = dx$

$du = \frac{dx}{x}$ and $v = x$

$$\int u\, dv = u\, v - \int v\, du$$

$$\int \ln x\, dx = x \ln x - \int x \times \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + c$$

$2.\int sin^{-1} 2x dx$

$$u = \sin^{-1} 2x \quad \text{and} \quad dv = dx$$

$$du = \frac{2dx}{\sqrt{1-4x^2}} \quad \text{and} \quad v = x$$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int \sin^{-1} 2x\,dx = x\sin^{-1} 2x - \int \frac{2xdx}{\sqrt{1-4x^2}}$$

$$= x\sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$$

$$\int \sin^n x \cos^m x$$

If m odd ⟹ $\cos^2 x = 1 - \sin^2 x$

If n odd ⟹ $\sin^2 x = 1 - \cos^2 x$

If n, m even ⟹

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \sin 2x)$$

2- Power of Trigonometric Functions

A. To Find the integral for: $\int \sin^n(x)\cos^m(x)$

case (1): If one of the powers is odd (or both are odd), then we use the following form;

$$\sin^2(x) + \cos^2(x) = 1$$

Case (2): If both of the powers are even, then we use one of the following forms:

$$\sin^2(x) = \frac{1}{2}(1 - \cos 2x) \quad \text{or} \quad \cos^2(x) = \frac{1}{2}(1 + \cos 2x)$$

Examples: Evaluate the following integrals:

1) $\int \sin^2(x)\cos^3(x)dx \qquad \left[n : \text{odd} \ \& \ m : \text{even} \longrightarrow case(1)\right]$

$$= \int \sin^2(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^2(x)\cos(x)\left(1 - \sin^2(x)\right)dx$$

$$= \int \underbrace{\sin^2(x)}_{u}\underbrace{\cos(x)dx}_{du} - \int \underbrace{\sin^4(x)}_{u}\underbrace{\cos(x)dx}_{du}$$

$$= \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

2) $\int \sin^3(x)\cos^3(x)dx \qquad \left[n : \text{odd} \;\&\; m : \text{odd} \;\longrightarrow\; case(1)\right]$

$$= \int \sin^3(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^3(x)\cos(x)\Big(1-\sin^2(x)\Big)dx$$

$$= \int \underbrace{\sin^3(x)}_{u}\underbrace{\cos(x)dx}_{du} - \int \underbrace{\sin^5(x)}_{u}\underbrace{\cos(x)dx}_{du}$$

$$= \boxed{\frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C}$$

B. To Find the integral for: $\boxed{\int \tan^n(x)\sec^m(x)}$

Case (1): If the powers sec(x) is even, we divide the sec(x) as:

$$\boxed{\sec^m(x) = \sec^{m-2}(x)\sec^2(x)} \;\&\; \text{use } \boxed{\sec^2(x) = 1+\tan^2(x)}$$

Case (2): If the powers tan(x) is odd, we divide the tan(x) as:

$$\boxed{\tan^n(x) = \tan^{n-1}(x)\tan(x)} \;\&\; \text{use } \boxed{\tan^2(x) = \sec^2(x)-1}$$

Examples: Evaluate the following integrals:

$$\int \sec^4(x)\tan^2(x)dx$$

$$= \int \sec^2(x)\sec^2(x)\tan^2(x)dx$$

$$= \int \Big(1+\tan^2(x)\Big)\sec^2(x)\tan^2(x)dx$$

$$= \int \underbrace{\tan^2(x)}_{u}\underbrace{\sec^2(x)dx}_{du} + \int \underbrace{\tan^4(x)}_{u}\underbrace{\sec^2(x)dx}_{du}\,dx$$

$$= \boxed{\frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C}$$

C. To find the integral for: $\int \cot^n(x)\csc^m(x)$

Case (1): If the power of $csc(x)$ is even, we divide the $csc(x)$ as:

$\boxed{\csc^m(x) = \csc^{m-2}(x)\csc^2(x)}$ & we use $\boxed{\csc^2(x) = 1 + \cot^2(x)}$

Case (2): If the power of $cot(x)$ is odd, we divide the $cot(x)$ as:

$\boxed{\cot^n(x) = \cot^{n-1}(x)\cot(x)}$ & we use $\boxed{\cot^2(x) = \csc^2(x) - 1}$

Examples: Evaluate the following integrals:

$$\int \csc^4(x)\cot^2(x)dx$$

$$= \int \csc^2(x)\csc^2(x)\cot^2(x)dx$$

$$= \int \left(1 + \cot^2(x)\right)\csc^2(x)\cot^2(x)dx$$

$$= \int \underbrace{\cot^2(x)}_{u}\underbrace{\csc^2(x)dx}_{du} + \int \underbrace{\cot^4(x)}_{u}\underbrace{\csc^2(x)dx}_{du}\,dx$$

$$= \boxed{-\frac{\cot^3(x)}{3} - \frac{\cot^5(x)}{5} + C}$$

2) $\int \frac{\cot^3(x)}{\sqrt{\csc(x)}}dx$

$$= \int \cot^2(x)\cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \left(\csc^2(x) - 1\right)\cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \csc^{\frac{3}{2}}(x)\cot(x)dx - \int \cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \underbrace{\csc^{\frac{1}{2}}(x)}_{u}\underbrace{\csc(x)\cot(x)dx}_{du} - \int \underbrace{\csc^{\frac{-3}{2}}(x)}_{u}\underbrace{\csc(x)\cot(x)(x)dx}_{du}$$

$$= \boxed{\frac{\csc^{\frac{3}{2}}(x)}{\frac{3}{2}} - \frac{\csc^{\frac{-1}{2}}(x)}{\frac{-1}{2}} + C}$$