The Limits

A limit is the value that a function or sequence "approaches" as the input or index approaches some value. We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x\to a} f(x) = L$$

Properties of the limit

1. If a and c are constants then $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$

Let $f_1(x) = L_1$ and $f_2(x) = L_2$ then

3. $\lim_{x\to a}\big(f_1(x) + f_2(x)\big) = L_1 + L_2$
4. $\lim_{x\to a}\big(f_1(x) - f_2(x)\big) = L_1 - L_2$
5. $\lim_{x\to a}\big(f_1(x).f_2(x)\big) = L_1.L_2$
6. $\lim_{x\to a}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$; $L_2 \neq 0$
7. If $\lim_{x\to a} g(x) = L$ then $\lim_{x\to a} f\big(g(x)\big) = f(L)$

Example 8: Find the following limits

1. $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3}\frac{1}{(\sqrt{x}+\sqrt{3})} = \frac{1}{2\sqrt{3}}$

2. $\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5}$

Divide top and bottom by x^2, then we get

$$\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5} = \lim_{x\to\infty}\frac{5+(3/x^2)}{7+(2/x)-(5/x^2)} = \frac{5}{7}$$

3. $\lim_{x\to\infty}\frac{(2x+1)^4}{(x^2+2x-3)^2} = \left(\lim_{x\to\infty}\frac{(2x+1)^2}{x^2+2x-3}\right)^2 = \left(\lim_{x\to\infty}\frac{4x^2+4x+1}{x^2+2x-3}\right)^2$

$$= \left(\lim_{x\to\infty}\frac{4+(4/x)+(1/x^2)}{1+(2/x)-(3/x^2)}\right)^2 = 4^2 = 16$$

4. $$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$=\lim_{x\to 0}\frac{4+x-4}{x\sqrt{4+x}+2}=\lim_{x\to 0}\frac{x}{x\sqrt{4+x}+2}=\frac{1}{4}$$

5.
$$\lim_{x\to\infty}\sqrt{x^2+x+1}-x=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)\times\frac{\left(\sqrt{x^2+x+1}+x\right)}{\left(\sqrt{x^2+x+1}+x\right)}$$

$$=\lim_{x\to\infty}\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}=\lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}$$

$$=\lim_{x\to\infty}\frac{\frac{x+1}{x}}{\frac{\sqrt{x^2+x+1}+x}{x}}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}=\frac{1}{2}$$

Derivatives

Rules for finding derivatives

1. Constant Rule

The derivative of a constant is always zero. That is if $f(x)=c$ then $f'(x)=0$.

2. Power Rule The derivative of a power function, $f(x)=x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π.

The derivative is $f'(x)=nx^{n-1}$

Example 1:

y	$\frac{dy}{dx}$
$y=x^4$	$\frac{dy}{dx}=4x^3$
$y=x^{-4}$	$\frac{dy}{dx}=-4x^{-5}=-\frac{4}{x^5}$
$y=\frac{1}{x^2}=x^{-2}$	$\frac{dy}{dx}=-2x^{-3}=-\frac{2}{x^3}$
$y=x^\pi$	$\frac{dy}{dx}=\pi x^{\pi-1}$

3. Multiplication by constant: The derivative of $cf(x)$ is $cf'(x)$

4. Sum Rule: The derivative of $f(x)+g(x)$ is $f'(x)+g'(x)$

5. Difference Rule: The derivative of $f(x)-g(x)$ is $f'(x)-g'(x)$

6. Product Rule: The derivative of the product of two functions is not the product of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx}\big(f(x)\times g(x)\big)=f(x)\times g'(x)+f'(x)\times g(x)$$

Example 2: Find derivative of the function $y=(3x^2+5)(2x^3-5x-4)$

$$\frac{dy}{dx}=(3x^2+5)\times(6x^2-5)+(2x^3-5x-4)\times 6x$$

7. Quotient Rule: *The derivative of the quotient of two functions is not the quotient of the functions' derivatives; rather, it is described by the equation below:*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\times f'(x)-f(x)\times g'(x)}{\big(g(x)\big)^2}$$

Example 3: Find derivative of the function

$$y=\frac{x^2-1}{x^2+1}$$

$$\frac{dy}{dx}=\frac{(x^2+1)\times 2x-(x^2-1)\times 2x}{(x^2+1)^2}=\frac{2x^3+2x-(2x^3-2x)}{(x^2+1)^2}$$

$$=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

8. Chain Rule:

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1.If we define $F(x)=(f\circ g)(x)$ then the derivative of $F(x)$ is,

$$F'(x)=f'\big(g(x)\big)g'(x)$$

2.If we have $y=f(u)$ and $u=g(x)$ then the derivative of y is,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Example If $y = u^2 - 2u$ and $u = \sqrt{3x+1}$, find $\dfrac{dy}{dx}$

$$\frac{dy}{du} = 2u - 2 = 2(u-1) \quad \text{and} \quad \frac{du}{dx} = \frac{3}{2\sqrt{3x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2(u-1) \times \frac{3}{2\sqrt{3x+1}} = \frac{3(u-1)}{\sqrt{3x+1}} = \frac{3\left(\sqrt{3x+1}-1\right)}{\sqrt{3x+1}}$$

Example 6: If $y = t + \dfrac{1}{t}$ and $x = t - \dfrac{1}{t}$, find $\dfrac{dy}{dx}$

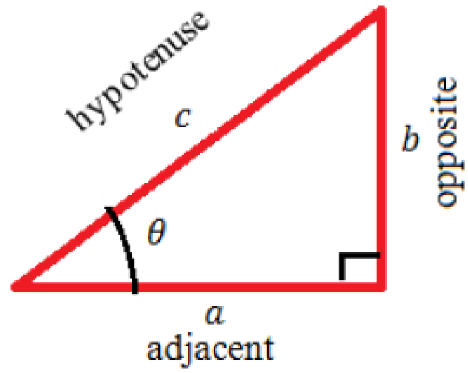
$$\frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2} \qquad \text{and} \qquad \frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2-1}{t^2} \times \frac{t^2}{t^2+1} = \frac{t^2-1}{t^2+1}$$

Trigonometric functions

Definitions of trigonometric functions for a right triangle

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.

For angle θ, the trigonometric functions are defined as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} \qquad , \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a} \quad , \quad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

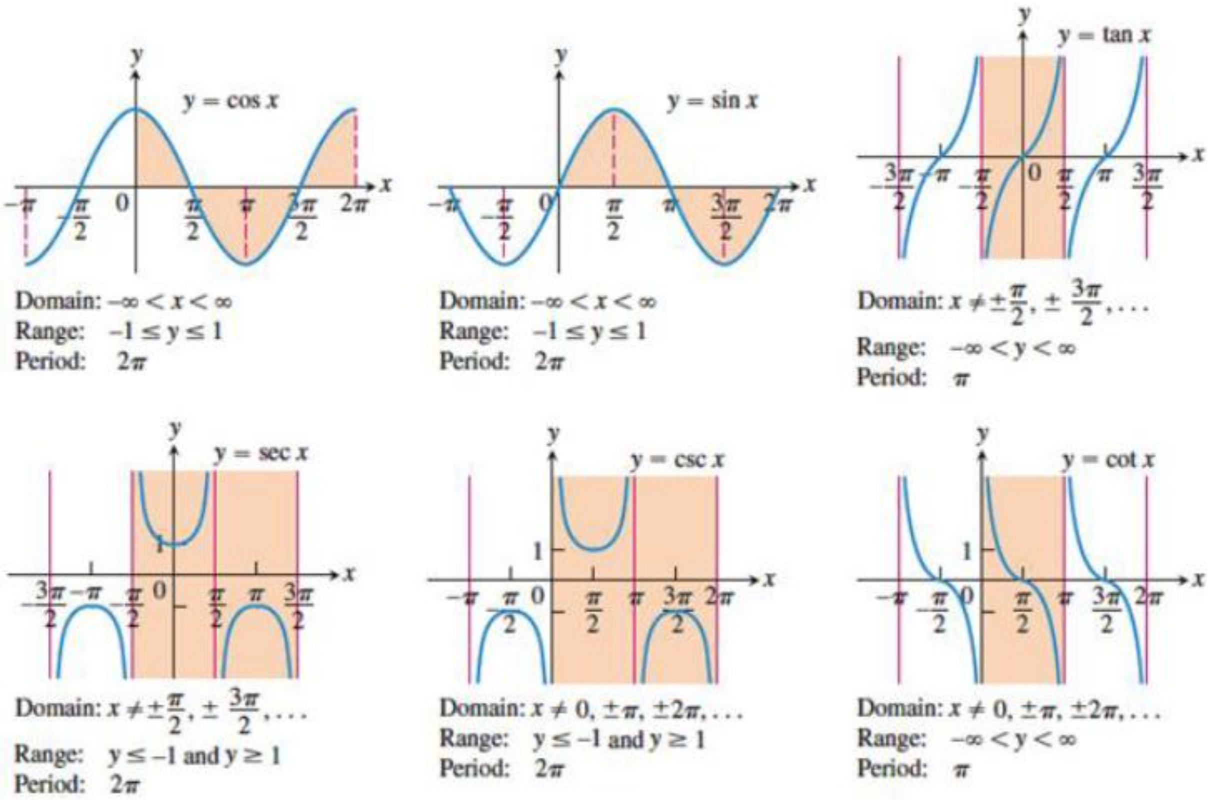
Trigonometric functions of negative angles

$$\sin(-\theta) = -\sin\theta \quad , \quad \cos(-\theta) = \cos\theta \quad \text{and} \quad \tan(-\theta) = -\tan\theta$$

Some useful relationships among trigonometric functions

1. $\sin^2 x + \cos^2 x = 1 \quad , \quad \sec^2 x - \tan^2 x = 1 \quad , \quad \csc^2 x - \cot^2 x = 1$
2. $\sin 2x = 2\sin x\cos x \ , \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
3. $\sin^2 x = \dfrac{1-\cos 2x}{2} \quad , \quad \cos^2 x = \dfrac{1+\cos 2x}{2}$

Graphs of Trigonometric Functions



Derivatives of trigonometric functions

If u is a function x, the chain rule version of this differentiation rule is

$$1.\ \frac{d}{dx}(\sin u) = \cos u\,.\frac{du}{dx}$$

$$2.\ \frac{d}{dx}(\cos u) = -\sin u\,.\frac{du}{dx}$$

$$3.\ \frac{d}{dx}(\tan u) = \sec^2 u.\frac{du}{dx}$$

$$4.\ \frac{d}{dx}(\cot u) = -\csc^2 u.\frac{du}{dx}$$

$$5.\ \frac{d}{dx}(\sec u) = \sec u \tan u\,.\frac{du}{dx}$$

$$6.\ \frac{d}{dx}(\csc u) = -\csc u \cot u\,.\frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \sin^2 x \quad \Rightarrow \quad y = (\sin x)^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2\sin x \cos x = \sin 2x$
2. $y = \cos(x^2) \quad \Rightarrow \quad \frac{dy}{dx} = -2x \sin(x^2)$
3. $y = \tan\sqrt{x} \quad \Rightarrow \quad \frac{dy}{dx} = \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$
4. $y = x^2 \sec 3x \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x = x \sec 3x\,(2 + 3x \tan 3x)$
5. $y = \sqrt{\sin 2x} \quad \Rightarrow \quad y = (\sin 2x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(\sin 2x)^{-1/2} \times \cos 2x \times 2$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example 2: If $y = \tan 2t$ and $x = \sec 2t$ show that $\frac{dy}{dx} = \csc 2t$

$$\frac{dy}{dt} = 2\sec^2 2t \quad , \quad \frac{dx}{dt} = 2\sec 2t \tan 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\sec^2 2t \times \frac{1}{2\sec 2t \tan 2t} = \frac{\sec 2t}{\tan 2t}$$

$$= \frac{\frac{1}{\cos 2t}}{\frac{\sin 2t}{\cos 2t}} = \frac{1}{\sin 2t} = \csc 2t$$

Derivative of natural logarithm function

If u is a function x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} . \frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \ln(5x + 1)$

$$\frac{dy}{dx} = \frac{1}{5x + 1} \times 5 = \frac{5}{5x + 1}$$

2. $y = 2x \tan^{-1} x - \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{2x}{1 + x^2} + 2 \tan^{-1} x - \frac{2x}{x^2 + 1} = 2 \tan^{-1} x$$

3. $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3 \cos 3x = 3 \cot 3x$$

4. $y = \ln(x^2 + 3)^{(x^2+3)}$ ⇨ $y = (x^2 + 3) \ln(x^2 + 3)$

$$\frac{dy}{dx} = (x^2 + 3) \times \frac{2x}{(x^2 + 3)} + 2x \ln(x^2 + 3) = 2x + 2x \ln(x^2 + 3)$$

Integral

If the function $F(x)$ is an antiderivative of $f(x)$ then the expression $F(x)+C$ where C is an arbitrary constant, is called the indefinite integral of $f(x)$ and we write

$$\int f(x)\,dx = F(x) + C$$

The integral of the power function

If u is a function of x, then

1. $\int u^n du = \dfrac{u^{n+1}}{n+1} + c \ ; n \in \mathbb{R} \ , \ n \neq -1$

2. $\int \dfrac{du}{u} = \ln|u| + c$

Examples

1. $\int \sqrt{3x+1}\,dx = \dfrac{1}{3}\int 3\,(3x+1)^{\frac{1}{2}} dx = \dfrac{1}{3}(3x+1)^{\frac{3}{2}} \times \dfrac{2}{3} + c = \dfrac{2}{9}(3x+1)^{\frac{3}{2}} + c$

2. $\int \dfrac{xdx}{\sqrt{x^2-3}} = \dfrac{1}{2}\int 2x(x^2-3)^{-1/2}dx = \dfrac{1}{2}(x^2-3)^{1/2} \times 2 + c = \sqrt{x^2-3} + c$

3. $\int \dfrac{xdx}{x^2-3} = \dfrac{1}{2}\ln|x^2-3| + c$

The integrals of trigonometric functions

If u is a function of x, then

1. $\int \sin u\,du = -\cos u + c$
2. $\int \cos u\,du = \sin u + c$
3. $\int \sec^2 u\,du = \tan u + c$
4. $\int \csc^2 u\,du = -\cot u + c$
5. $\int \sec u \tan u\,du = \sec u + c$
6. $\int \csc u \cot u\,du = -\csc u + c$
7. $\int \tan u\,du = -\ln|\cos u| + c = \ln|\sec u| + c$
8. $\int \cot u\,du = \ln|\sin u| + c = -\ln|\csc u| + c$

Methods of integration

I- Integration by parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated. For example, if the problem is to find

$$\int x^2 \cos x\, dx$$

We have two way to do this:

1- Tabular integration by parts

Which is used for

1) the products of polynomials and sine function, $x^n \sin bx$.

2) the products of polynomials and cosine function, $x^n \cos bx$.

3) the products of polynomials and exponential function, $x^n e^{ax}$.

In any of these three cases we choose the polynomial as u and the product of sine function and dx (cosine or exponential function and dx – respectively) as dv.

Example1: Evaluate

$$1. \int x^3 \cos x\, dx \qquad 2. \int x^2 e^{2x}\, dx$$

	Derivatives of u		Integrals of v
$1. \int x^3 \cos x\, dx$	x^3	$+$	$\cos x$
	$3x^2$	$-$	$\sin x$
	$6x$	$+$	$-\cos x$
	6	$-$	$-\sin x$
	0		$\cos x$

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$$

2. $\int x^2 e^{2x}\,dx$

$D.of\ x^2$	$I.of\ e^{2x}$
x^2 (+)	e^{2x}
$2x$ (−)	$\frac{1}{2}e^{2x}$
2 (+)	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \frac{2x}{4}e^{2x} + \frac{2}{8}e^{2x} + c = \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + c$$

2- Integration by using the formula

$$\int u\,dv = u\,v - \int v\,du$$

Which is used for inverse trigonometric functions and logarithm functions.

Example2: Evaluate

$$\int \ln x\,dx$$

$u = \ln x$ and $dv = dx$

$du = \frac{dx}{x}$ and $v = x$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int \ln x\,dx = x\ln x - \int x \times \frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

$$\int x \sec x \tan x\,dx$$

$u = x$ $du = dx$

$dv = \sec x \tan x\,dx$ $v = \sec x$

$$\int x \sec x \tan x\,dx = x \sec x - \int \sec x\,dx$$

$$= x \sec x - \ln|\sec x + \tan x| + c$$