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# *Al-Mustaqbal University College*
# *Air Conditioning and Refrigeration Techniques Engineering Department*

# Strength of Materials

## Second Stage


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## ( 2020 – 2021 )

# Torsion

## Introduction

- In this chapter we will concentrate on the behavior of slender members subjected to torsional loading that is, loading by couples that produce twisting of the member about its axis.
- Figure (4.1) shows a common example of torsional loading and indicates the shear stresses and the stress resultant associated with torsion.
- In Figure (4.1), shaft AB is say a ***torsion member***.
- The shaft AB is subjected to equal and opposite torques of magnitude ***T*** that twist one end relative to the other.

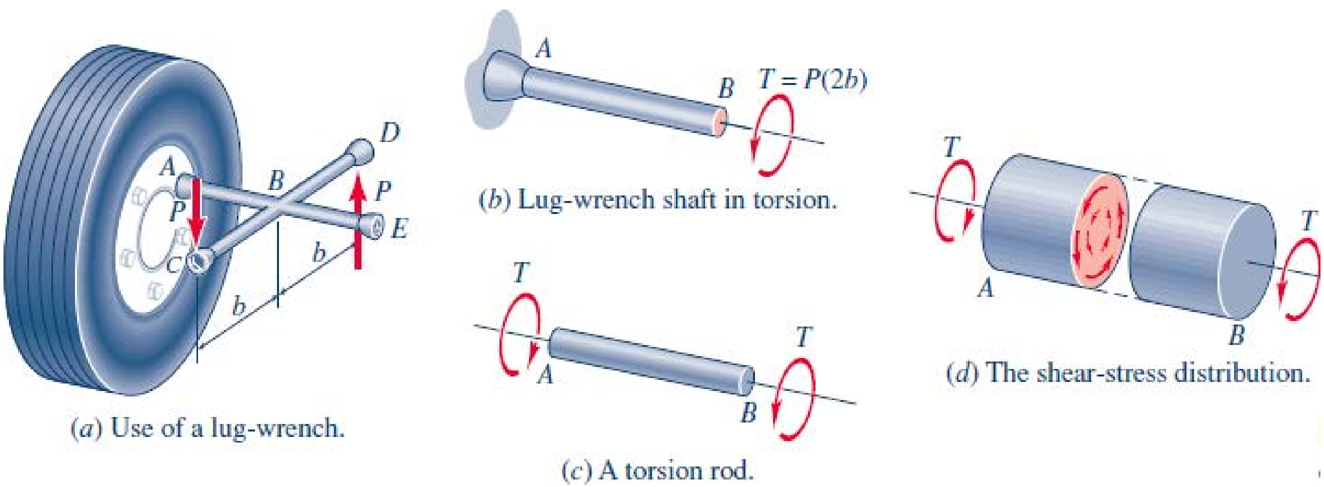



**Figure (4.1): An example of torsion**

- Several names are applied to torsion members, depending on the application:
  - *shaft*,
  - *torque tube*,
  - *torsion rod*,
  - *Torsion bar*, or
  - *Simply torsion member*.

## ❑ Analogy Between Axial Deformation and Torsion

▶ There is a direct analogy between axial deformation and torsion, as indicated by the entries in Table 4.1.

**TABLE 4.1 Analogy Between Axial Deformation and Torsion**

| Axial Deformation | Torsion |
|---|---|
| Axial Force ($F$) | Torque ($T$) |
| Elongation ($e$) | Twist angle ($\phi$) |
| Normal stress ($\sigma$) | Shear stress ($\tau$) |
| Extensional strain ($\epsilon$) | Shear strain ($\gamma$) |
| Modulus of elasticity ($E$) | Shear modulus ($G$) |

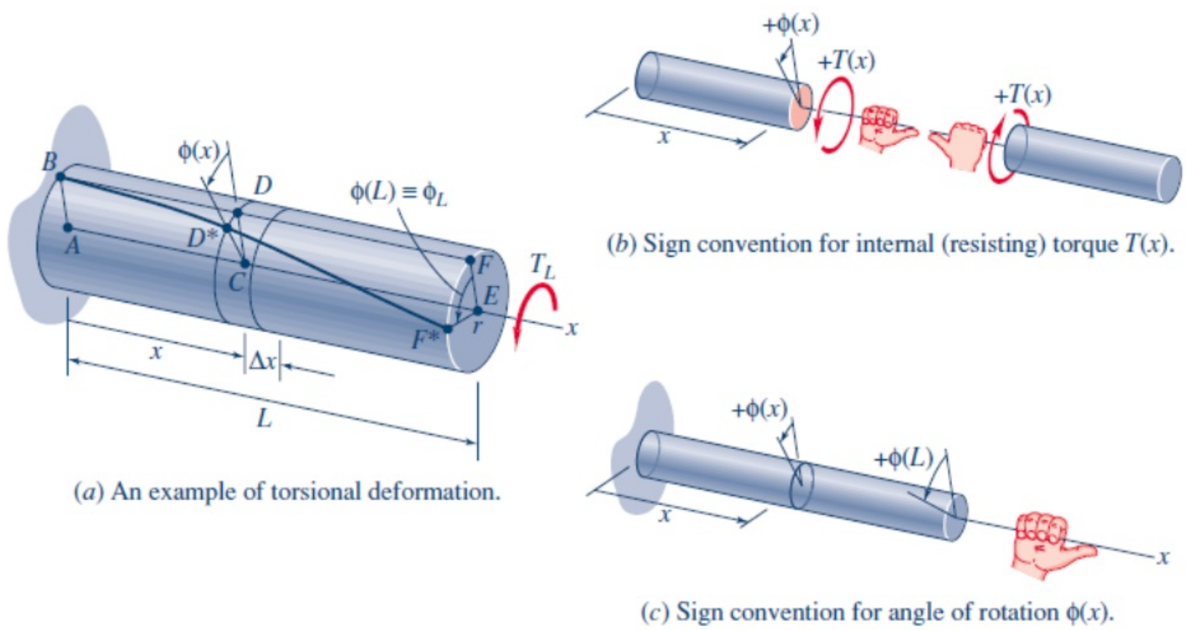
## <u>Sign convention</u>

**A sign convention for torsion** is defined as follows:

▶ The longitudinal axis of the bar is labeled the x axis, with one end of the member being taken as the origin.

▶ A positive torque, ***T(x)***, is a moment that acts on the cross section at ***x*** in a right-hand-rule sense about the outer normal to the cross section. On a cross sectional cut at ***x*** there will be equal and opposite torques ***T(x)***, as indicated in Figure (4.2b).

▶ A positive angle of rotation, $\emptyset(\boldsymbol{x})$**,** is a rotation of the cross section at ***x*** in <u>**a right-hand-rule sense**</u> about the ***x*** axis, as illustrated in Figure (4.2c).




(*a*) An example of torsional deformation.

(*b*) Sign convention for internal (resisting) torque $T(x)$.

(*c*) Sign convention for angle of rotation $\phi(x)$.

**Figure (4.2): Torsional deformation; Sign convention for torsion**

## Assumptions

Following assumptions are made, while finding out shear stress in a circular shaft subjected to torsion:

1) The material of the shaft is uniform,

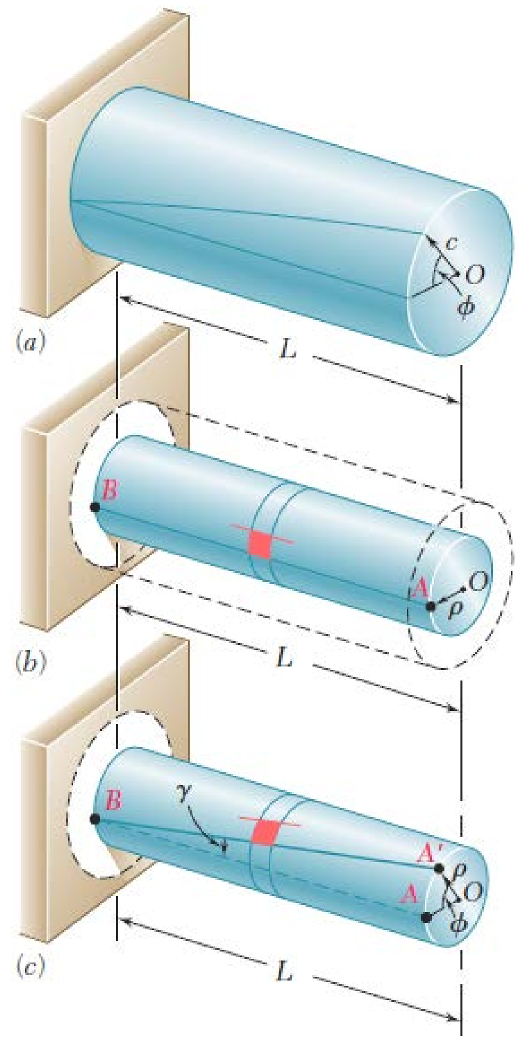
2) The twist along the shaft is uniform,

3) Normal cross – sections of the shaft, which were plane and circular before the twist, remain plane and circular even after the twist, and

4) All diameters of the normal cross – section, which were straight before the twist, remain straight with their magnitude unchanged, after the twist.

## Strain-Displacement Analysis

To determine the distribution of ***shearing strains*** in a circular shaft of length $\boldsymbol{L}$ and radius $\boldsymbol{c}$ that has been twisted through an angle $\emptyset$ (Figure 4.3*a*).




- Detaching from the shaft a cylinder of radius $\boldsymbol{r}$, we consider the small square element before any load is applied (Figure 4.3b).
- As the shaft is subjected to a torsional load, the element deforms into a rhombus (Figure 4.3*c*).
- The shearing strain $\boldsymbol{\gamma}$ in a given element is measured by the change in the angles formed by the sides of that element.
- Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain $\boldsymbol{\gamma}$ must be equal to the angle between lines AB and A'B.
- We observe from Figure (4.3c) that, for small values of $\boldsymbol{\gamma}$, we can express the arc length AA' as:

$$AA' = \gamma L$$

- But, on the other hand, we have:

$$AA' = \rho\emptyset$$

- It follows that:

$$\rho\emptyset = \gamma L$$

Or

$$\gamma = \frac{\rho\emptyset}{L} \qquad \text{... (4.1)}$$

Where $\boldsymbol{\gamma}$ and $\emptyset$ are both expressed in radians.

- It follows from Eq. (4.1) that the shearing strain is maximum on the surface of the shaft, where $\boldsymbol{\rho} = \boldsymbol{c}$. We have:

$$\gamma_{max} = \frac{c.\emptyset}{L} \qquad \text{... (4.2)}$$

- Eliminating $\emptyset$ from Eqs. (4.1) and (4.2), we can express the shearing strain g at a distance r from the axis of the shaft as:

$$\gamma = \frac{\rho}{c}\gamma_{max} \qquad \text{... (4.3)}$$

## STRESSES IN THE ELASTIC RANGE

- Let us now consider the case when the torque ***T*** is such that all shearing stresses in the shaft remain below the yield strength $\tau_y$.

- Recalling Hooke's law for shearing stress and strain:

$$\tau = G\gamma \qquad \text{... (4.4)}$$

Where ***G*** is the modulus of rigidity or shear modulus of the material.

Multiplying both members of Eq. (4.3) by ***G***, we write:

$$G\gamma = \frac{\rho}{c}G\gamma_{max} \qquad \text{... (4.5)}$$

$$\tau = \frac{\rho}{c}\tau_{max} \qquad \text{... (4.6)}$$

- Figure (4.4*a*) shows the stress distribution in a solid circular shaft of radius ***c***, and Figure (4.4*b*) in a hollow circular shaft of inner radius $c_1$ and outer radius $c_2$. From Eq. (4.6), we find that, in the latter case,

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} \qquad \text{... (4.7)}$$

- The sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude ***T*** of the torque exerted on the shaft:

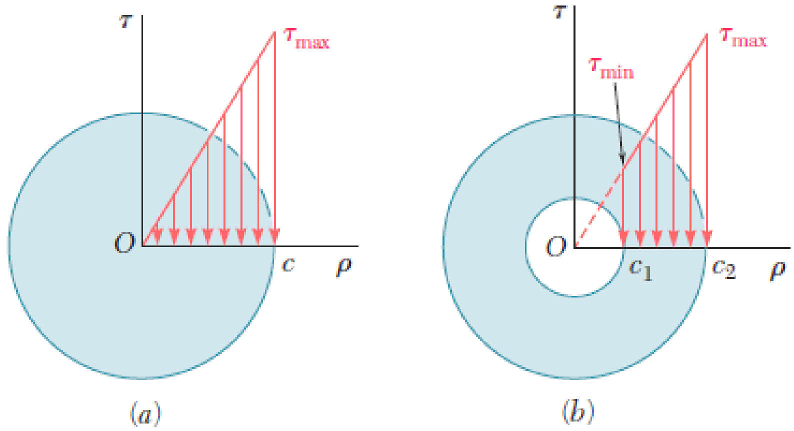



**Figure (4.4): Distribution of shearing stresses.**

$$\int \rho(\tau dA) = T \qquad \text{... (4.8)}$$

Substituting for $\boldsymbol{\tau}$ from (4.6) into (4.8), we write:

$$T = \int \rho(\tau dA) = \frac{\tau_{max}}{c}\int \rho^2\, dA \qquad \text{... (4.9)}$$

But, $\int \rho^2\, dA = J\ (\,Polar\ moment\ of\ unertia\mathbf{:})$, then:

$$T = \frac{\tau_{max}.J}{c} \qquad \text{... (4.10)}$$

$$or \qquad \boldsymbol{\tau_{max}} = \frac{\boldsymbol{T.c}}{\boldsymbol{J}} \qquad \text{... (4.11)}$$

- Substituting for $\boldsymbol{\tau_{max}}$ from Eqs.(4.11) into (4.6), we express the shearing stress at any distance $\boldsymbol{\rho}$ from the axis of the shaft as:

$$\boldsymbol{\tau} = \frac{\boldsymbol{T\rho}}{\boldsymbol{J}} \qquad \text{... (4.12)}$$

Equations (4.11) and (4.12) are known as ***the elastic torsion formulas***.

- Substituting for ($\boldsymbol{\gamma}$) from Eq. (4.1) into Eq. (4.4) and then substituting for ($\boldsymbol{\tau}$) from Eq. (4.4) into Eq.(4.12), we express the angle of twist (∅) in term of the torque (***T***):

$$\emptyset = \frac{Tl}{GJ} \qquad \text{... (4.13)}$$

Recall from statics that the polar moment of inertia of a circle of radius (***r***) is:

$$J = \frac{1}{2}\pi r^4$$

- In the case of a hollow circular shaft of inner radius ($c_1$) and outer radius ($c_2$), the polar moment of inertia is:

$$J = \frac{1}{2}\pi c_2^4 - \frac{1}{2}\pi c_1^4 = \frac{1}{2}\pi(c_2^4 - c_1^4)$$

- In Eq. 4.13, the torque-twist equation, *T*, *G*, and *J* may each vary with ***x*** as shown in Figure (4.5). This derivative form of the torque-twist equation may be integrated over the length of the member to give:

$$\emptyset = \int_0^L \frac{T(x).dx}{G(x).J(x)} \qquad \text{... (4.14)}$$

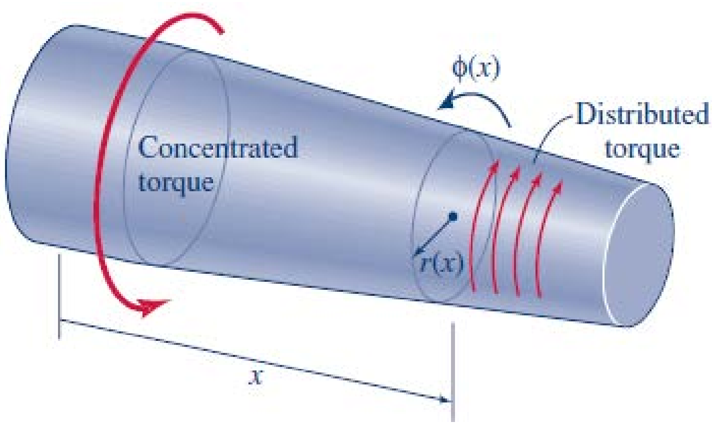



**Figure (4.5)**

## Examples

**Example (4.1):** A uniform shaft of radius ***r*** and length ***L*** is subjected to a uniform distributed external torque $t_0$ (moment per unit length). (See Figure 4.6)

(a) Determine an expression for the maximum shear stress $\tau_{max}$.

(b) Determine an expression for the total twist angle $\emptyset = \emptyset_L$.

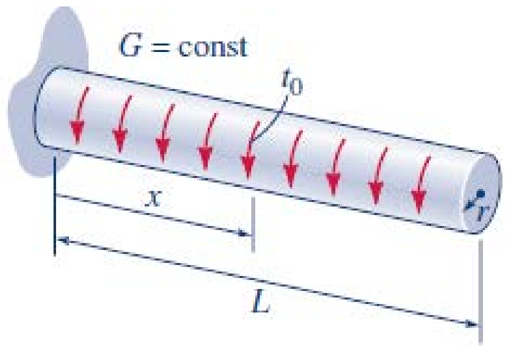
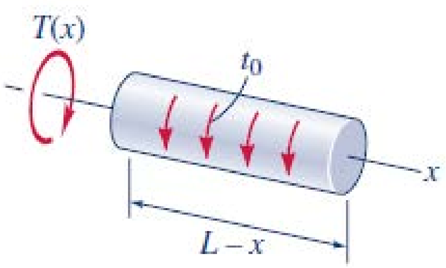



**Figure (4.6)**

**Solution:**

**a)**




**Equilibrium:** On the section at ***x***, the internal torque ***T*(*x*)** in the positive sense according to the right-hand rule is:

$$\sum M_x = 0$$

$$T(x) = t_o(L - x)$$

**Shear Stress:** The maximum shear stress occurs at ***x* = *0***, where $T(0) = T_{max} = t_o(L)$.Then, from the torsion formula, Eq. 4.11,

$$\tau_{max} = \frac{t_o(L).r}{\frac{\pi}{2}r^4} = \frac{2t_o(L)}{\pi r^3}$$

**b)**

***Torque-Twist:*** From the torque-twist relationship, Eq. 4.14,

$$\emptyset_L = \int_0^L \frac{t_o(L-x)\,.dx}{G.J} = \frac{t_o}{G.J}\int_0^L (L-x)\,.dx$$

$$\emptyset_L = \frac{t_o}{G.J}\left[Lx - \frac{x^2}{2}\right]_0^L = \frac{t_o.L^2}{2G.J} = \frac{t_o.L^2}{G.\pi.r^4}$$

**Example (4.2):** A hollow cylindrical steel shaft is **1.5 m** long and has inner and outer diameters respectively equal to **40** and **60 mm** (Figure 4.7).

**(*a*)** What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed **120 MPa**?

**(*b*)** What is the corresponding minimum value of the shearing stress in the shaft?

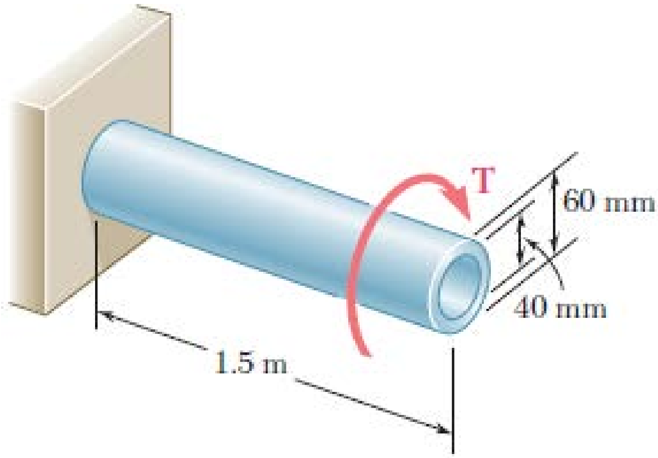



**Figure (4.7)**

**Solution:**

**(a) Largest Permissible Torque.**

The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max}$ =**120** MPa. Since this value is less than the yield strength for steel, we can use Eq. (4.11). Solving this equation for ***T***, we have:

$$\tau_{max} = \frac{T.c}{J} \rightarrow T = \frac{\tau_{max}.J}{c}$$

$$J = \frac{1}{2}\pi c_2^4 - \frac{1}{2}\pi c_1^4 = \frac{1}{2}\pi(c_2^4 - c_1^4)$$

$$c_1 = \frac{40}{2} = 20\ mm\ (\ or\ 0.02\ m)$$

$$c_2 = \frac{60}{2} = 30\ mm\ (\ or\ 0.03\ m)$$

$$J = \frac{1}{2}\pi((0.03)^4 - (0.02)^4) = 1.021 \times 10^{-6}\ m^4$$

$$T = \frac{\tau_{max}.J}{c} = \frac{(120 \times 10^6\ Pa)(1.021 \times 10^{-6}\ m^4)}{0.03} = 4.08\ kN.m$$

**(b) Minimum Shearing Stress.** The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (4.7), which expresses that $\tau_{min}$and $\tau_{max}$are respectively proportional to $c_1$ and $c_2$:

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max}$$

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} = \frac{0.02}{0.03}(120) = 80\ MPa$$

**Example (4.3):** What is the minimum diameter of a solid steel shaft that will not twist through more than ($3^o$) in a (6 m) length when subjected to a torque of (**14 kN.m**)? What maximum shearing stress is developed? Use **G = 83 GPa**.

**Solution:**

$$\emptyset = \frac{Tl}{GJ} \rightarrow 3\left(\frac{\pi}{180}\right) = \frac{(14 \times 10^3)(6)}{(83 \times 10^9)\left(\frac{\pi}{2}r^4\right)}$$

$$\rightarrow r = 59.23\ mm\ or\ d = 118.5\ mm$$

$$\tau_{max} = \frac{Tr}{J} = \frac{14 \times 10^6(59.23)}{\frac{\pi}{2}(59.23)^4} = 42.9\ MPa$$

**Example (4.4):** Determine the length of the shortest (**2mm**) diameter bronze wire which can be twisted through two complete turns without exceeding a shearing stress of (**70 MPa**). Use **G = 83 GPa**.

**Solution:**

$$J = \frac{\pi}{32}d^4 = \frac{\pi}{32}(2)^4 = 1.571\ mm^4$$

$$\tau_{max} = \frac{T.c}{J} \rightarrow T = \frac{\tau_{max}.J}{c} \rightarrow$$

$$T = \frac{70\ (\pi(1)^4)}{2(1)} \rightarrow T \cong 110\ N.mm$$

$$\emptyset = \frac{Tl}{GJ} \rightarrow 720\left(\frac{\pi}{180}\right) = \frac{(110)(l)}{(83 \times 10^3)(1.571)}$$

$$\rightarrow l = 14895\, mm \;\; or\; 14.895\, m$$

**Example (4.5):** A compound shaft consisting of an aluminum segment and a steel segment is acted upon by two torque as shown in Figure (4.8). Determine the maximum permissible value of (T) subjected to the following conditions ($\boldsymbol{\tau_s \le 100\ MPa}$ , $\boldsymbol{\tau_{al} \le 70\ MPa}$, and the angle of rotation of the free end limited to (**12º**). Use ***G<sub></sub>***$\boldsymbol{G_s}$ **= 83 GPa** and $\boldsymbol{G_{al}}$ **= 28 GPa**.

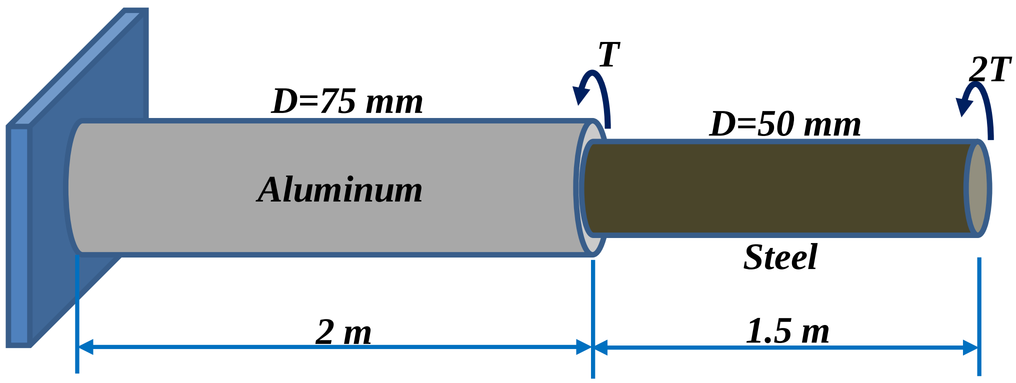


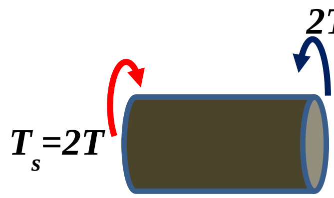

**Figure (4.8)**

**Solution:**

$$\tau_s = \frac{T.c}{J} \rightarrow T = \frac{\tau_s.J}{c} \rightarrow T_s = \frac{\tau_s(\pi c^4)}{2.c} = \frac{\tau_s(\pi c^3)}{2} = \frac{100\ (\pi(25)^3)}{2} \times 10^{-3}$$
$$= 2454.37\ N.m$$

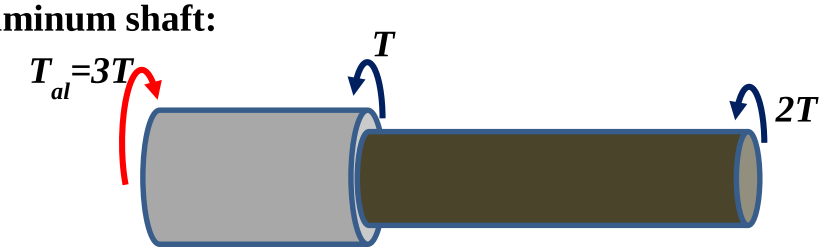
$$T_{al} = \frac{\tau_s(\pi c^3)}{2} = \frac{70\ (\pi(37.5)^3)}{2} \times 10^{-3} = 5798.45\ N.m$$

**Section in steel shaft:**




$$T_s = 2T \;\rightarrow T = \frac{T_s}{2} = \frac{2454.37}{2} = 1227.2\ N.m$$

**Section in aluminum shaft:**




$$T_{al} = 3T \rightarrow T = \frac{T_{al}}{3} = \frac{5798.45}{3} = 1932.82\ N.m$$

$$\emptyset_{free} = \emptyset_s + \emptyset_{al}$$

$$12\left(\frac{\pi}{180}\right) = \frac{2T(1.5)}{(83\times10^9)\frac{\pi}{32}(50)^4\times10^{-12}} + \frac{3T(2)}{(83\times10^9)\frac{\pi}{32}(75)^4\times10^{-12}}$$

$$12 = 7.328\times10^{-3}T \rightarrow T = 1637.53\ N.m$$

***Use T = 1227.2 N.m***

**<u>Example (4.6):</u>** For the shaft shown in Figure (4.9), find the maximum shear stress and total angle of rotation. Assume G=83 GPa and diameter of shaft (25 mm).

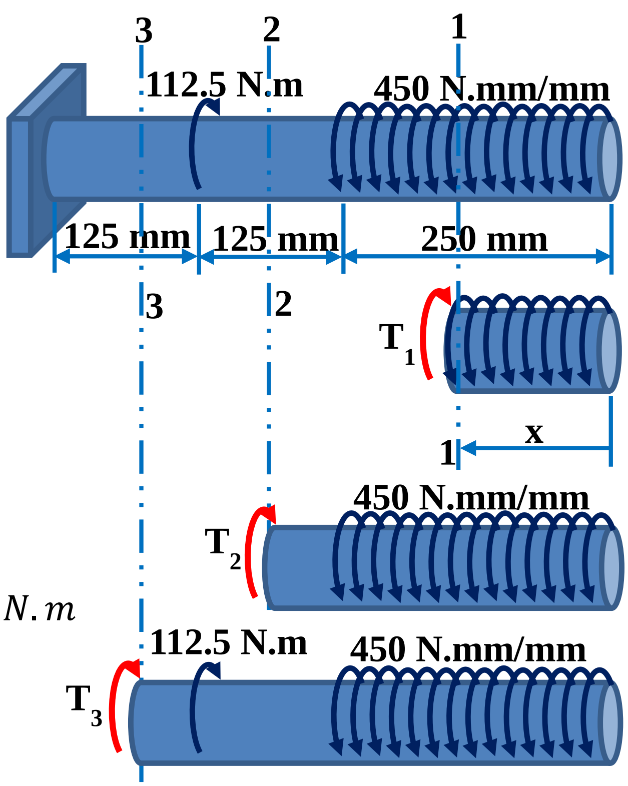



**Figure (4.9)**

**<u>Solution:</u>**

**Sec.(1-1) as F.B.D.**

$$T = 450x$$

**Sec.(2-2) as F.B.D.**

$$T = 450(250)$$

$$T = 112500\ N.mm\ \ or\ 112.5\ N.m$$

**Sec.(3-3) as F.B.D.**

$$T + 112.5 = 450(250)\times10^{-3}$$

$$\rightarrow T = 0$$

$$\therefore T = 112.5\ N.m \qquad (Control)$$

$$\therefore \tau_{max} = \frac{T.c}{J} = \frac{(112.5 \times 10^3)(12.5)}{\frac{\pi}{2}(12.5)^4} = 36.67\ MPa$$

$$\emptyset_{total} = \sum \emptyset_i = 0 + \emptyset_1 + \emptyset_2$$

$$\emptyset_1 = \int_0^{250} \frac{(450x)dx}{GJ} = \frac{450x^2}{2GJ}\Big]_0^{250} = \frac{450(250)^2}{2(84 \times 10^3)\left(\frac{\pi}{32}(25)^4\right)}$$

$$\emptyset_1 = 4.365 \times 10^{-3}\ rad.$$

$$\emptyset_2 = \frac{112.5 \times 10^3 (125)}{(84 \times 10^3)\left(\frac{\pi}{32}(25)^4\right)} = 4.365 \times 10^{-3}\ rad.$$

$$\emptyset_{total} = [4.365 \times 10^{-3} + 4.365 \times 10^{-3}]\frac{180}{\pi} = 0.5^o$$

**Example (4.7):** Find the total angle of rotation for the prismatic shaft shown in Figure (4.10), which is subjected to distributed torsional moment (***T*<sub></sub>**$T_{(x)}=kx$ N.mm/mm), ***GJ*** and ***k*** are constants.

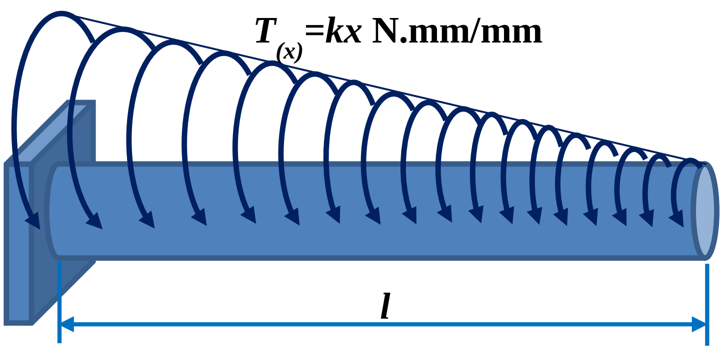
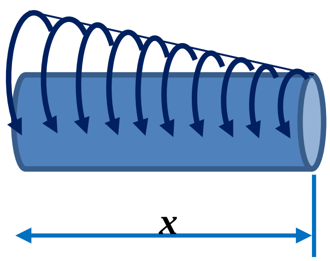



**Figure (4.10)**

**Solution:**

$$T = \int_0^x (kx)dx = \frac{kx^2}{2}$$



$$\emptyset = \int_0^l \frac{T}{GJ} dx = \frac{1}{GJ} \int_0^l \frac{kx^2}{2} dx$$

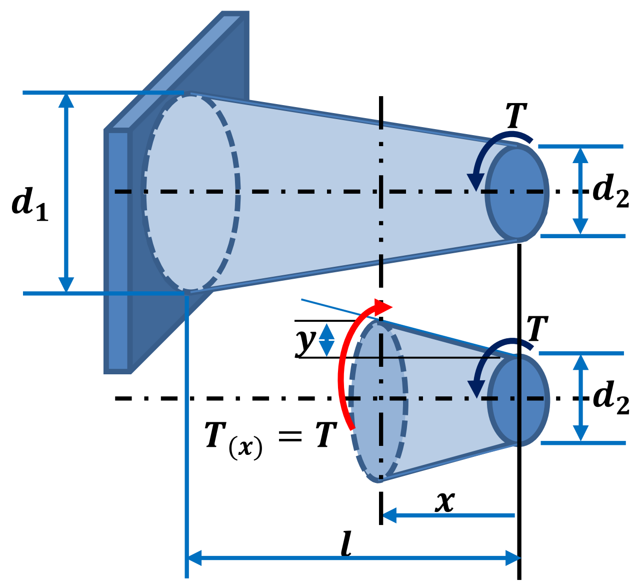
$$\therefore \emptyset = \frac{1}{2GJ} \left[ \frac{kx^3}{3} \right]_0^l = \frac{kl^3}{6GJ}$$

**<u>To find max. Shear stress:</u>**

$$\tau_{max} \frac{16T}{\pi d^3} = \frac{16kx^2}{2\pi d^3} \bigg]_{x=l} = \frac{16kl^2}{2\pi d^3} = \frac{8kl^2}{\pi d^3}$$

**<u>Example (4.8):</u>** For the non – prismatic shaft shown in Figure (4.11). Find the total rotation.

**Figure (4.11)**




**<u>Solution:</u>**

$$\frac{\frac{d_1 - d_2}{2}}{l} = \frac{y}{x}$$

$$\rightarrow y = \frac{d_1 - d_2}{2l} x$$

$$\emptyset = \int_0^l \frac{T_{(x)} dx}{GJ_{(x)}}$$

$$J_{(x)} = \frac{\pi}{32} (D_{(x)})^4$$

$$D_{(x)} = 2y + d_2 = \frac{d_1 - d_2}{l} x + d_2$$

$$J_{(x)} = \frac{\pi}{32} \left[ \frac{d_1 - d_2}{l} x + d_2 \right]^4$$

$$\emptyset = \int_0^l \frac{T dx}{G \frac{\pi}{32} \left[\frac{d_1 - d_2}{l} x + d_2\right]^4} = \frac{32T}{G\pi} \int_0^l \left[\frac{d_1 - d_2}{l} x + d_2\right]^{-4} dx$$

Multiply by $\frac{\frac{d_1-d_2}{l}}{\frac{d_1-d_2}{l}}$, get:

$$\emptyset = \frac{32T}{G\pi(d_1 - d_2)} \left[\left[\frac{d_1 - d_2}{l} x + d_2\right]^{-3} . \frac{1}{-3}\right]_0^l$$

$$= \frac{32T}{-3G\pi(d_1 - d_2)} \left[(d_1\ )^{-3} - (d_2\ )^{-3}\right]$$

$$\emptyset = \frac{32T}{3G\pi} \left[\frac{(d_1\ )^2 + d_1 d_2 + (d_2\ )^2}{(d_1\ )^3 (d_2\ )^3}\right]$$