Solution of first order differential equation

المحاضره الثانيه لتكملة المحاضره الاولى

الطريقه الثالثه

بعد ان شرحنا الطريقه الاولى والثانيه في المحاضره الاولى اليوم سنشرح الطريقه الثالثه والرابعه

ادناه الطريقه الثالثه

Homogenous differential equations –by substituting y=vx

المعادله التاليه لا نستطيع فصل ال y عن ال x

Here is an equation:

$$\frac{dy}{dx} = \frac{x + 3y}{2x}$$

This looks simple enough, but we find that we cannot express the RHS in the form of '*x*-factors' and '*y*-factors', so we cannot solve by the method of separating the variables.

In this case we make the substitution $y = vx$, where v is a function of x. So $y = vx$. Differentiate with respect to x (using the product rule):

$$\therefore \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Also $$\frac{x+3y}{2x} = \frac{x+3vx}{2x} = \frac{1+3v}{2}$$

The equation now becomes $v + x\frac{dv}{dx} = \frac{1+3v}{2}$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+3v}{2} - v$$

$$= \frac{1+3v-2v}{2} = \frac{1+v}{2}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+v}{2}$$

The given equation is now expressed in terms of v and x, and in this form we find that we can solve by separating the variables. Here goes:

$$\int \frac{2}{1+v}dv = \int \frac{1}{x}dx$$

$$\therefore \quad 2\ln(1+v) = \ln x + C = \ln x + \ln A$$

$$(1+v)^2 = Ax$$

But $y = vx \quad \therefore \quad v = \left\{\frac{y}{x}\right\} \quad \therefore \quad \left(1+\frac{y}{x}\right)^2 = Ax$

which gives $(x+y)^2 = Ax^3$

Note: $\frac{dy}{dx} = \frac{x+3y}{2x}$ is an example of a *homogeneous differential equation.*

This is determined by the fact that the total degree in x and y for each of the terms involved is the same (in this case, of degree 1). The key to solving every homogeneous equation is to substitute $y = vx$ where v is a function of x. This converts the equation into a form which we can solve by separating the variables.

تتلخص هذه الطريقه بالخطوات التاليه:

1. ناخذ الجانب الايسر من المعادله dy/dx ونعوض y=vx ونشتق (dy/dx =(الاول xمشتقة الثاني + الثانيxمشتقة الاول) وتصبح (v.1 +x dv/dx) على اعتبار ان ال v داله لل x .

2. ونعوض نفس الشيء في الجانب الايمن ونكمل الحل كما اعلاه بالتسلسل وبهذا يكمل الحل

Example 2:

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Here, all terms of the RHS are of degree 2, i.e. the equation is homogeneous.
$\therefore$ We substitute $y = vx$ (where v is a function of x)

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

هذه ثابته

and $\frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v}$

The equation now becomes:

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$= \frac{1 + v^2 - v^2}{v} = \frac{1}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1}{v}$$

Now you can separate the variables and get the result in terms of v and x.

$$\boxed{\frac{v^2}{2} = \ln x + C}$$

Because

$$\int v\,dv = \int \frac{1}{x}dx$$

$$\therefore \frac{v^2}{2} = \ln x + C$$

All that remains now is to express v back in terms of x and y. The substitution we used was $y = vx$ $\therefore$ $v = \frac{y}{x}$

$$\therefore \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + C$$

$$y^2 = 2x^2(\ln x + C)$$

Now, what about this one?

Example 3:

Solve $\frac{dy}{dx} = \frac{2xy + 3y^2}{x^2 + 2xy}$

$$y = vx, \text{ where } v \text{ is a function of } x$$

Right. That is the key to the whole process.

$$\frac{dy}{dx} = \frac{2xy + 3y^2}{x^2 + 2xy}$$

So express each side of the equation in terms of v and x:

$$\frac{dy}{dx} = \dots\dots\dots\dots$$

and $\dfrac{2xy + 3y^2}{x^2 + 2xy} = \dots\dots\dots\dots$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{2xy + 3y^2}{x^2 + 2xy} = \frac{2vx^2 + 3v^2x^2}{x^2 + 2vx^2} = \frac{2v + 3v^2}{1 + 2v}$$

So that $v + x\dfrac{dv}{dx} = \dfrac{2v + 3v^2}{1 + 2v}$

Now take the single v over to the RHS and simplify, giving:

$$x\frac{dv}{dx} = \dots\dots\dots\dots$$

$$\begin{aligned} x\frac{dv}{dx} &= \frac{2v + 3v^2}{1 + 2v} - v \\ &= \frac{2v + 3v^2 - v - 2v^2}{1 + 2v} \\ x\frac{dv}{dx} &= \frac{v + v^2}{1 + 2v} \end{aligned}$$

Now you can separate the variables, giving …………

$$\int \frac{1 + 2v}{v + v^2}\, dv = \int \frac{1}{x}\, dx$$

Integrating both sides, we can now obtain the solution in terms of v and x. What do you get?

$$\ln(v + v^2) = \ln x + C = \ln x + \ln A$$

$$\therefore\ v + v^2 = Ax$$

We have almost finished the solution. All that remains is to express v back in terms of x and y.

Remember the substitution was $y = vx$, so that $v = \dfrac{y}{x}$

So finish it off.

$$\boxed{xy + y^2 = Ax^3}$$

Because

$v + v^2 = Ax$ and $v = \frac{y}{x}$ $\quad \therefore \frac{y}{x} + \frac{y^2}{x^2} = Ax$

$xy + y^2 = Ax^3$

And that is all there is to it.

الطريقه الرابعه

Linear equation –by using integration factor:

لنأخذ المعادله التاليه التى لا يمكن بالطرق السابقه ولهذا تعتبر هذه الطريقه الانسب:

Consider the equation $\frac{dy}{dx} + 5y = e^{2x}$

This is clearly an equation of the first order, but different from those we have dealt with so far. In fact, none of our previous methods could be used to solve this one, so we have to find a further method of attack.

المعادله اعلاه كالشكل التالي

dy/dx + py =Q

p and Q are function of x

integration factor الان معادلة ال

كما يلي

I.F= $e^{\int pdx}$

p=5 في المعادله

So, $\int pdx = 5x$

So, I.F= e^{5x}

I.F x نضرب طرفي المعادله

dy/dx. $e^{5x} + 5ye^{5x} = e^{7x}$

dy/dx (ye^{5x}) **نرى ان الجانب الايسر عباره عن**

So,

$$\therefore \frac{d}{dx}\left\{y.e^{5x}\right\} = e^{7x}$$

Integrate both side with respect to x

$$y.e^{5x} = \int e^{7x}\,dx = \frac{e^{7x}}{7} + C$$

$$\boxed{y = \frac{e^{2x}}{7} + Ce^{-5x}}$$

Example 2:

To solve $\frac{dy}{dx} - y = x$

If we compare this with $\frac{dy}{dx} + Py = Q$, we see that in this case

$P = -1$ and $Q = x$.

The integrating factor is always $e^{\int P dx}$ and here $P = -1$.

$\therefore \int P\,dx = -x$ and the integrating factor is therefore ....

So, the I.F= $e^{\int pdx}$

I.F= e^{-x}

We therefore multiply both sides by e^{-x}.

$$\therefore\ e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

$$\frac{d}{dx}\left\{e^{-x}y\right\} = xe^{-x} \quad \therefore\ ye^{-x} = \int xe^{-x}\,dx$$

The RHS integral can now be determined by integrating by parts:

$$ye^{-x} = x(-e^{-x}) + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\therefore\ y = -x - 1 + Ce^{x} \quad \therefore\ y = Ce^{x} - x - 1$$

The whole method really depends on:

(a) being able to find the integrating factor

(b) being able to deal with the integral that emerges on the RHS.

Let us consider the general case.

اذن نستنج القانون التالى للحل وهذا مهم جدا:

$$y.\text{IF} = \int Q.\text{IF}\,dx$$

IF= INTEGRATION FACTOR

EXAMPLE 3:

Solve $x\frac{dy}{dx}+y=x^3$

First we divide through by x to reduce the first term to a single $\frac{dy}{dx}$

i.e. $\frac{dy}{dx}+\frac{1}{x}.y=x^2$

Compare with $\left[\frac{dy}{dx}+Py=Q\right]$ $\therefore\ P=\frac{1}{x}$ and $Q=x^2$

$IF=e^{\int P dx}$ $\int P\,dx=\int\frac{1}{x}\,dx=\ln x$

$\therefore\ IF=e^{\ln x}=x$ $\therefore\ IF=x$

The solution is $y.IF=\int Q.IF\,dx$

so $yx=\int x^2.x\,dx=\int x^3\,dx=\frac{x^4}{4}+C$ $\therefore\ xy=\frac{x^4}{4}+C$

انتهت المحاضره الثانيه