Ministry of Higher Education and Scientific Research

Al-Mustaqbal University College

Department of Chemical Engineering and petroleum Industrials

Week: 1,2,3

Mathematics II

2nd Stage

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1. Double integral

The definite integral can be extended to functions of more than one variable. Consider, for example, a function of two variables z=f (x,y).The double integral of function f (x,y)is denoted by

$$\iint_R F(x,y)dA$$

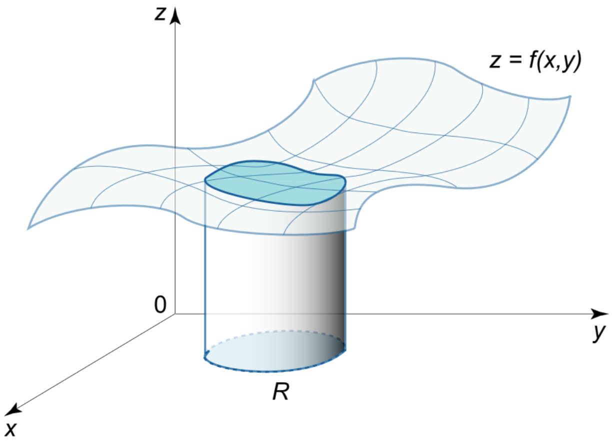


Figure 1

Where R is the region of integration in the xy-plane.

The definite integral $\int_a^b f(x)dx$ of a function of one variable f(x)≥0 is the area under the curve f(x) from x=a to x=b, then the double integral is equal to the volume under the surface z=f (x,y) and above the xy-plane in the region of integration R (Figure 1).

a- Properties of double integral

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$$

4. ***Additivity:*** $$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

if R is the union of two nonoverlapping regions R_1 and R_2

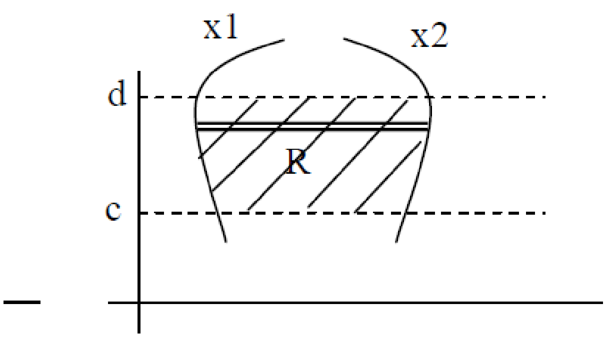
b- **Cartesian form**

Double integral of $f(x, y)$over the region R is denoted by:

$$\iint_R F(x,y)dA = \iint_R F(x,y)\, dx\, dy = \int_c^d \int_{x1}^{x2} F(x,y)\, dx\, dy$$ Fig.2a

or

$$\iint_R F(x,y)dA = \iint_R F(x,y)\, dy\, dx = \int_a^b \int_{y1}^{y2} F(x,y)\, dy\, dx$$ Fig.2b



(a) (b)

Figure 2

C- **Finding Limits of Integration in cartesian form**

- **Using Vertical Cross-Sections**

When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1- Sketch. Sketch the region of integration and label the bounding curves.(Figure 3 a).

2- Find the y-limits of integration. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants)(Figure 3 b).

3- Find the x-limits of integration. Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure 3 c) is

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

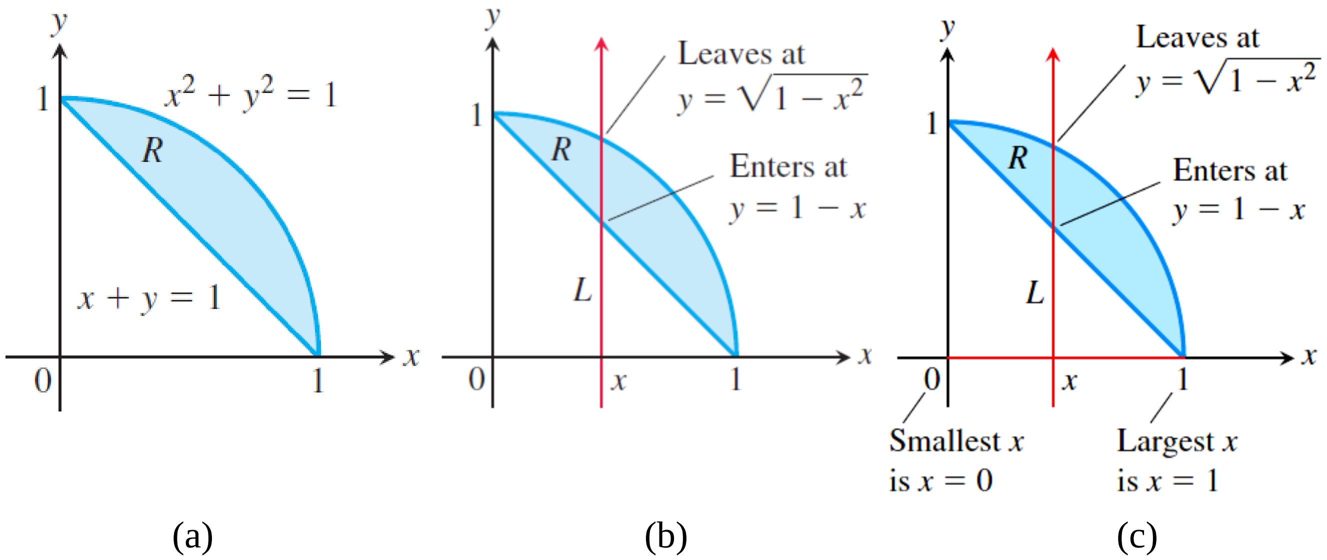


Figure 3

- **Using Horizontal Cross-Sections**

To evaluate the same double integral as an iteratedintegral with the order of integration reversed, use horizontal lines instead of vertical linesin Steps 2 and 3(see Figure 4). The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

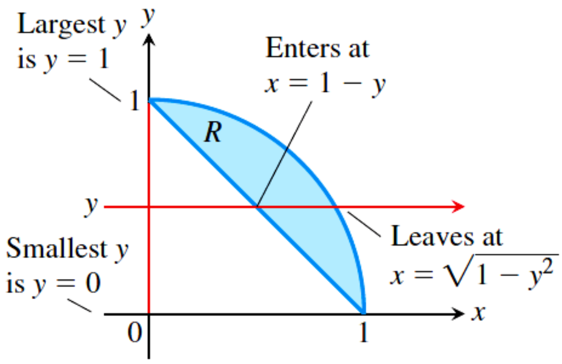


Figure 4

d- **Polar form**

$$\iint_R F(r,\theta)\, dA = \int_{r_1}^{r_2} \int_{\theta_1=g_1(r)}^{\theta_2=g_2(r)} F(r,\theta)\ r\, d\theta\, dr$$

or

$$\iint_R F(r,\theta)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1=g_1(\theta)}^{r_2=g_2(\theta)} F(r,\theta)\ r\, dr\, d\theta$$

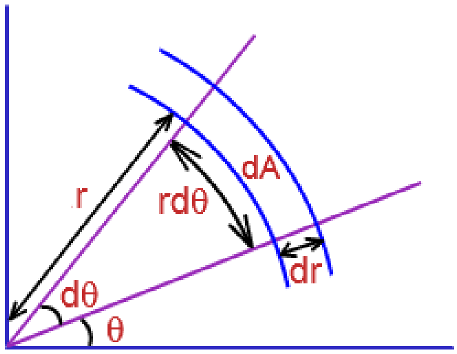


Figure 5

e- **Finding Limits of Integration in polar form**

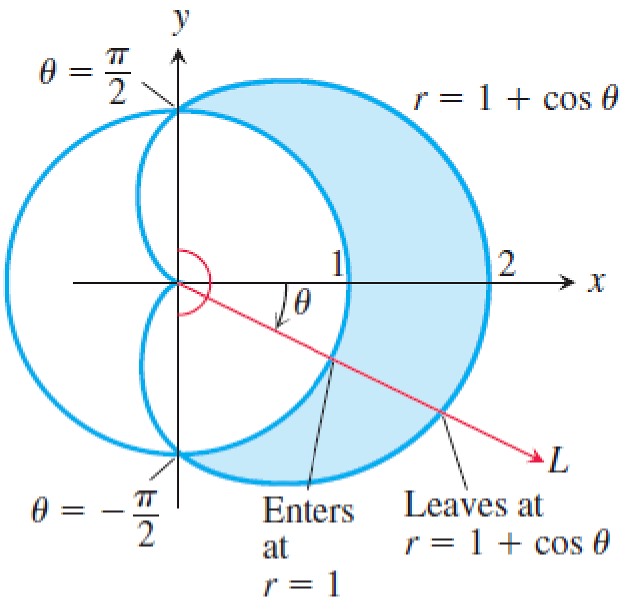
The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r,\theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1- *Sketch*. Sketch the region and label the bounding curves.

2- *Find the r-limits of integration*. Imagine a ray *L* from the origin cutting through *R* in thedirection of increasing *r*. Mark the *r*-values where *L* enters and leaves *R*. These are the*r*-limits of integration. They usually depend on the angle u that *L* makes with the positive*x*-axis.

3- *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound *R*.These are the θ-limits of integration (see figure 6). The polar iterated integral is

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\,r\,dr\,d\theta.$$



$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta.$$

Figure 6

f- **Change of variables**

Let $x = x\,(u,v)\,, y = y\,(u,v)$ then the formula for a change of variables in double integrals from x, y to u, v is

$$\iint_R F(x,y)\,dy\,dx = \iint_{R'} F(x\,(u,v)\,, y\,(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

that is, the integrand is expressed in terms of u and v, and dx, dy is replaced by du dv times

the absolute value of the Jacobian.

$$j = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

For double integral transformation from the cartesian coordinates to polar coordinates ordinates as follows:

$$\text{Since} \qquad x = r\cos\theta \quad , \quad y = r\sin\theta$$

using the Jacobian matrix, we find that

$$j = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r((\cos\theta)^2 + (\sin\theta)^2) = r$$

Then

$$\int_{x1}^{x2}\int_{y1}^{y2} F(x,y)\,dy\,dx = \int_{r1}^{r2}\int_{\theta 1}^{\theta 2} F(r,\theta)\,r\,dr\,d\theta$$