# Integral

## Indefinite Integration

**General Indefinite Integration:** In calculus, an antiderivative of a function f(x) is a differentiable function F(x) whose derivative is equal to the original function f(x).

$$F'(x) = f(x) \implies \frac{dF(x)}{dx} = f(x)$$

$$\implies dF(x) = f(x)dx$$

$$\overset{\int}{\implies} \int dF(x) = \int f(x)dx$$

$$\implies \boxed{F(x) = \int f(x)dx + C}, \text{ where } C \in \mathbb{R}.$$

**Example:** $y' = 2x \implies \frac{dy}{dx} = 2x$

$$\implies dy = 2x\ dx \overset{\int}{\implies} \int dy = \int 2x\ dx \implies \boxed{y = \frac{x^2}{2} + C}$$

### Indefinite Integrals Properties:

Let $f(x)$ be a function and $x \in \mathbb{R}$, then:

i. $\int af(x) = a\int f(x)dx$, where $a$ is a constant.

ii. $\int \Big(f(x) \mp g(x)\Big)dx = \int f(x)dx \mp \int g(x)dx$

**Notes:**

- $\int \Big(f(x) * g(x)\Big)dx \neq \int f(x)dx * \int g(x)dx$
- $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

**Rules:** In general,

i. $\int dx = x + c$

ii. $\int u^n du = \frac{u^{n+1}}{n+1} + C$, where $C \in \mathbb{R}$.

**Examples:** Evaluate the following integrals:

1) $\int 3dx$

$= 3\int dx$

$= 3x + C$

2) $\int 5x^2dx$

$= 5\int x^2dx$

$= 5\frac{x^3}{3} + C$

3) $\int \frac{3\pi}{x^5}dx$

$= \int 3\pi x^{-5}dx$

$= 3\pi \int x^{-5}dx$

$= 3\pi \frac{x^{-4}}{-4} + C$

$= \frac{-3\pi}{4x^4} + C$

4) $\int (2x+3)dx$

$= \int 2xdx + \int 3dx$

$= x^2 + C_1 + 3x + C_2$

$= x^2 + 3x + C$

5) $\int \sqrt{2x+1}dx$

$= \frac{1}{2}\int (2x+1)^{\frac{1}{2}}.2dx$

$= \frac{1}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{1}{3}\sqrt{(2x+1)^3} + C$

6) $\int \frac{x^3-2x^7}{5x^5}dx$

$= \int \frac{x^3}{5x^5} - \frac{2x^7}{5x^5}dx$

$= \frac{1}{5}\int x^{-2}dx - \frac{2}{5}\int x^2dx$

$= \frac{1}{5}\frac{x^{-1}}{-1} - \frac{2}{5}\frac{x^3}{3} + C$

$= \frac{-1}{5x} - \frac{2}{15}x^3 + C$

7) $\int \frac{(z+1)dz}{\sqrt[3]{z^2+2z+2}}$

$= \int (z^2+2z+2)^{-\frac{1}{3}}(z+1)dz$

$= \frac{1}{2}\int (z^2+2z+2)^{-\frac{1}{3}}2(z+1)dz$

$= \frac{1}{2}\frac{(z^2+2z+2)^{\frac{2}{3}}}{\frac{2}{3}} + C$

$= \frac{3}{4}\sqrt[3]{(z^2+2z+2)^2} + C$

## Integrals of Trigonometric Functions

If $u$ is a function of $x$, then

1. $\int \sin u\, du = -\cos u + c$
2. $\int \cos u\, du = \sin u + c$
3. $\int \sec^2 u\, du = \tan u + c$
4. $\int \csc^2 u\, du = -\cot u + c$
5. $\int \sec u \tan u\, du = \sec u + c$
6. $\int \csc u \cot u\, du = -\csc u + c$
7. $\int \tan u\, du = -\ln|\cos u| + c = \ln|\sec u| + c$
8. $\int \cot u\, du = \ln|\sin u| + c = -\ln|\csc u| + c$

**Examples:** Evaluate the following integrals:

1) $\int \sin(3x)dx$

$= \frac{1}{3}\int \sin(3x).3dx$

$= \boxed{-\frac{1}{3}\cos(3x) + C}$

2) $\int \cos(2t)dt$

$= \frac{1}{2}\int \cos(2t).2\, dt$

$= \boxed{\frac{1}{2}\sin(2t) + C}$

3) $\int x\sec^2(x^2)dx$

$= \frac{1}{2}\int \sec^2(x^2).2xdx$

$= \boxed{\frac{1}{2}\tan(x^2) + C}$

4) $\int \cot(5x)\csc(5x)dx$

$= \frac{1}{5}\int \cot(5x)\csc(5x$

$= \boxed{-\frac{1}{5}\csc(5x) + C}$

**Remark:** Sometimes we should do some algebra to evaluate the integral.

**Examples:** Evaluate the following integrals:

1) $\int \sec^3(x)\tan(x)dx$

$$= \int \underbrace{\sec^2(x)}_{u}\underbrace{\sec(x)\tan(x)dx}_{du}$$

$$= \frac{\sec^3(x)}{3} + C$$

2) $\int \frac{\cos(2x)}{\sin^3(2x)}dx$

$$= \frac{1}{2}\int \underbrace{(\sin(2x))^{-3}}_{u}\underbrace{\cos(2x).2dx}_{du}$$

$$= \frac{1}{2}\frac{(\sin(2x))^{-2}}{-2} + C$$

$$= -\frac{1}{4\sin^2(2x)} + C$$

$$\int \frac{6-\cos(3x)}{\sin^2(3x)}dx$$

$$= \int \frac{6}{\sin^2(3x)}dx - \int \frac{\cos(3x)}{\sin^2(3x)}dx$$

$$= 6\int \frac{1}{\sin^2(3x)}dx - \int \frac{1}{\sin(3x)}\frac{\cos(3x)}{\sin(3x)}dx$$

$$= 6\int \csc^2(3x)dx - \int \csc(3x)\cot(3x)dx$$

$$= \frac{6}{3}\int \csc^2(3x).3dx - \frac{1}{3}\int \csc(3x)\cot(3x).$$

$$= 2(-\cot(3x)) - \frac{1}{3}(-\csc(3x)) + C$$

$$= \boxed{-2\cot(3x) + \frac{1}{3}\csc(3x) + C}$$

## Remark

**A.** When the power of $\sin(x)$ or $\cos(x)$ is **odd**, we use:

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

**B.** When the power of $\sin(x)$ or $\cos(x)$ is **even**, we use:

$$\boxed{\sin^2(x) = \frac{1}{2}(1 - \cos 2x)} \text{ or } \boxed{\cos^2(x) = \frac{1}{2}(1 + \cos 2x)}$$

**Examples:** Evaluate the following integrals:

1) $\int \sin^3(x)dx$

$$= \int \sin(x)\sin^2(x)dx$$

$$= \int \sin(x)(1 - \cos^2(x))dx$$

$$= \int \sin(x)dx - \int \underbrace{\cos^2(x)}_{u}\underbrace{\sin(x)dx}_{du}$$

$$= \boxed{\cos(x) - \frac{\cos^3(x)}{3} + C}$$

3) $\int \cos^4(x)dx$

$$= \int \left(\cos^2(x)\right)^2 dx$$

$$= \int \left(\tfrac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \tfrac{1}{4}\int \left((1+2\cos(2x)+\cos^2(2x))\right) dx$$

$$= \tfrac{1}{4}\int 1dx + \tfrac{1}{4}\int 2\cos(2x)dx + \tfrac{1}{4}\int \cos^2(2x)dx$$

$$= \tfrac{1}{4}\int dx + \tfrac{1}{2}\int \cos(2x)dx + \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos 4x)dx$$

$$= \tfrac{1}{4}\int dx + \tfrac{1}{2}.\tfrac{1}{2}\int \cos(2x).2dx + \tfrac{1}{8}\int dx + \tfrac{1}{8}.\tfrac{1}{4}\int \cos 4x.4dx$$

$$= \boxed{\frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C}$$

**Remark**

**A.** If the powers of $\tan(x)$ & $\cot(x)$ is even, we use:

use $\boxed{\tan^2(x) = \sec^2(x) - 1}$

**B.** If the powers of $\sec(x)$ & $\csc(x)$ is even, we use:

use $\boxed{\sec^2(x) = 1 + \tan^2(x)}$

**Examples:** Evaluate the following integrals:

**1)** $\int \tan^2(3x)dx$

$$= \int (\sec^2(3x) - 1)dx$$

$$= \tfrac{1}{3}\sec^2(3x).3dx - \int dx$$

$$= \boxed{\frac{1}{3}\tan(3x) - x + C}$$

2) $\int \sec^4(x)dx$

$$= \int \sec^2(x)\sec^2(x)dx$$

$$= \int \left(1+\tan^2(x)\right)\sec^2(x)dx$$

$$= \int \sec^2(x)dx + \int \tan^2(x)\sec^2(x)dx$$

$$= \boxed{\tan(x) + \frac{\tan^3(x)}{3} + C}$$

3) $\int \csc^4(x)dx$

$$= \int \csc^2(x)\csc^2(x)dx$$

$$= \int \left(1+\cot^2(x)\right)\csc^2(x)dx$$

$$= \int \csc^2(x)dx + \int \cot^2(x)\csc^2(x)dx$$

$$= \boxed{-\cot(x) - \frac{\cot^3(x)}{3} + C}$$

**Integrals of Inverse Trigonometric Functions:** We can derive all the integration forms from our derivatives forms as follows:

(1) $\frac{d}{du}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C}$

(2) $\frac{d}{du}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = -\cos^{-1}(u) + C}$

(3) $\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = \tan^{-1}(u) + C}$

(4) $\frac{d}{du}\cot^{-1}(u) = -\frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = -\cot^{-1}(u) + C}$

(5) $\frac{d}{du}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = \sec^{-1}(u) + C}$

(6) $\frac{d}{du}\csc^{-1}(u) = -\frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = -\csc^{-1}(u) + C}$

**Examples:** Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1-4x^2}}$

$= \frac{1}{2}\int \frac{2dx}{\sqrt{1-(2x)^2}}$

$= \boxed{\frac{1}{2}\sin^{-1}(2x)+C}$ or $\boxed{\frac{1}{2}\cos^{-1}(2x)+C}$

2) $\int \frac{dt}{1+t^2}$

$= \boxed{\tan^{-1}(t)+C}$ or $\boxed{-\cot^{-1}(t)+C}$

3) $\frac{dx}{x\sqrt{4x^2-1}}$

$= \frac{2dx}{2x\sqrt{(2x)^2-1}}$

$= \boxed{\sec^{-1}|2x|+C}$ or $\boxed{-\csc^{-1}|2x|+C}$

4) $\frac{-dx}{\sqrt{4-25x^2}}$

$= \frac{-dx}{\sqrt{4(1-\frac{25}{4}x^2)}}$

$= \frac{-dx}{2\sqrt{1-(\frac{5}{2}x)^2}}$

$= \frac{-1}{2}.\frac{2}{5}.\frac{\frac{5}{2}dx}{\sqrt{1-(\frac{5}{2}x)^2}}$

$= \boxed{-\frac{1}{5}\sin^{-1}(\frac{5}{2}x)+C}$ or $\boxed{\frac{1}{5}\cos^{-1}(\frac{5}{2}x)+C}$

5) $\int \frac{\cos(x)dx}{\sqrt{1-\sin^2(x)}}$

$= \sin^{-1}(\sin(x))+C$

$= \boxed{x+C}$

6) $\int \frac{\tan^{-1}(x)}{1+x^2}dx$

$= \int \tan^{-1}(x).\frac{dx}{1+x^2}$

$= \boxed{\frac{(\tan^{-1}(x))^2}{2}+C}$