Al-Mustaqbal University College
Air Conditioning and Refrigeration Techniques Engineering Department

Strength of Materials

Second Stage

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(2020 – 2021)

HOOKE'S LAW FOR PLANE STRESS

❑ THE RELATIONSHIP BETWEEN E AND G:

For a ***uniaxial stress state***, $\boldsymbol{\sigma_x}$, (Figure 6.7) the linear relationship between stress and strain, ϵ_x , was given by Hooke's Law:

$$\epsilon_x = \frac{\sigma_x}{E}$$

Poisson's ratio, ***v***, relates the transverse strain, ϵ_y, to ϵ_x:

$$\epsilon_y = \epsilon_z = -v\epsilon_x = -v\frac{\sigma_x}{E}$$

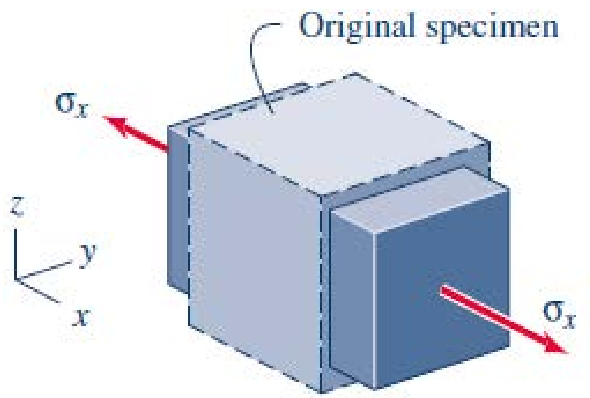


Figure 6.7

For linearly elastic materials, the shear modulus ***G*** and Young's modulus ***E*** are related by the equation:

$$G = \frac{E}{2(1+v)}$$

- For this equation to apply, the material must not only be linearly elastic, it must also be **isotropic,** and that is, *its material properties like **E** and **v** must be independent of orientation in the body.*

❑ **Plane Stress.** *A body that is subjected to a two-dimensional state of stress with* $\boldsymbol{\sigma_z = \tau_{xz} = \tau_{yz} = 0}$, *is said to be in a state of* **plane stress.** An element in plane stress is shown in Figure (6.8).

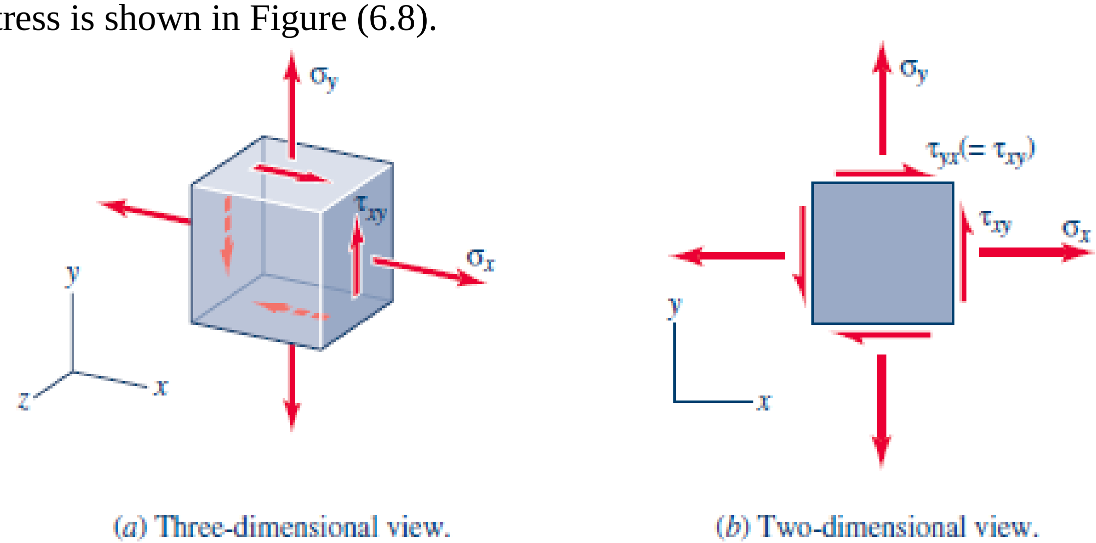


Figure (6.8)

If the material of which the body is composed is linearly elastic and isotropic, the effects of stresses $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$, and $\boldsymbol{\tau_{xy}}$ can be superposed, giving *Hooke's Law for plane stress:*

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - v\sigma_y\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - v\sigma_x\right)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Generalized Hooke's Law for Isotropic Materials

Let the body be subjected to stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz} as shown in Figure (6.9).

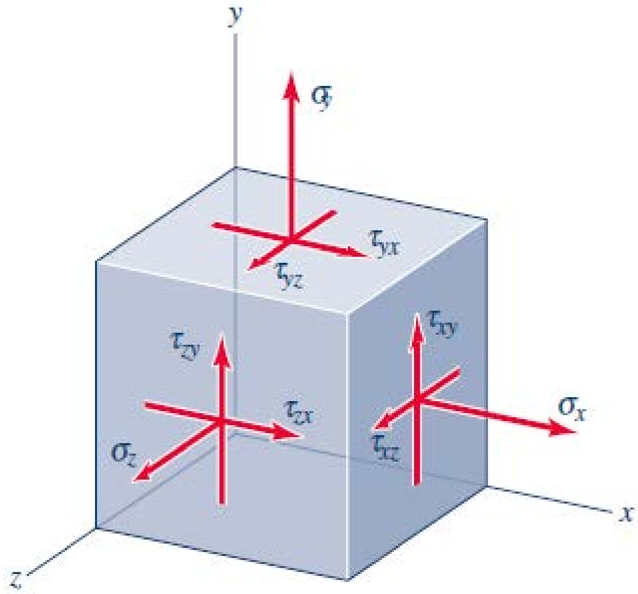


Figure (6.9)

Figure (6.10) illustrate the strains produced separately by the three normal stresses, σ_x, σ_y, and σ_z.

By the superposition principle, the total extensional strains are given by:

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - v\left(\sigma_y + \sigma_z\right)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - v(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - v\left(\sigma_x + \sigma_y\right)\right]$$

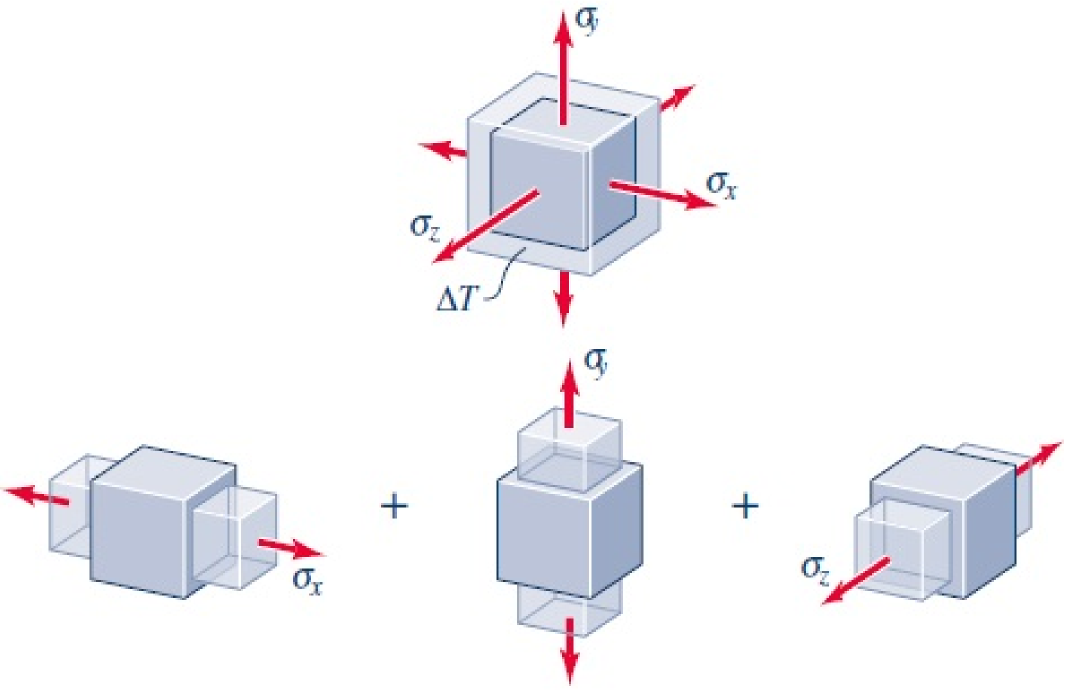


Figure (6.10)

- For an isotropic linearly elastic material, the shear stresses are related to the shear strains by the following equations:

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \;\; \gamma_{xz} = \frac{1}{G}\tau_{xz}, \;\; \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

These shear strains are illustrated in Figure (6.11).

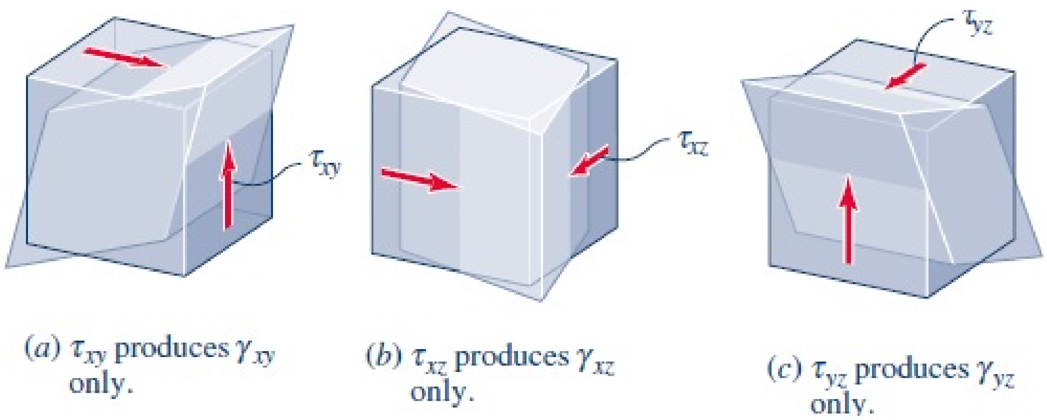


Figure (6.11): Illustration of shear strains.

Solving the above equations for the stresses in terms of the strains, we get:

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[(1-v)\epsilon_x + v\left(\epsilon_y + \epsilon_z\right)\right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[(1-v)\epsilon_y + v(\epsilon_x + \epsilon_z)\right]$$

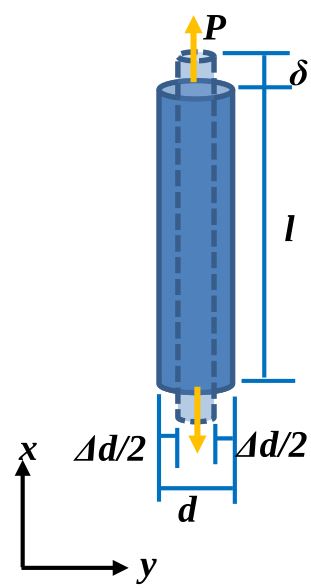
$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[(1-v)\epsilon_z + v\left(\epsilon_y + \epsilon_x\right)\right]$$

and

$$\tau_{xy} = G\gamma_{xy}\,,\ \ \tau_{xz} = G\gamma_{xz},\ \ \tau_{zy} = G\gamma_{zy}$$

Examples

Example (6.5): A prismatic bar of circular cross – section is loaded by tensile force (P=85 kN). The bar has length (3 m) and diameter (30 mm). It is made of aluminum with modulus of elasticity (70 GPa) and Poisson's ratio (1/3). Calculate the elongation, the decrease in diameter and the change in volume.



Solution:

$$\delta = \varepsilon_x . l \quad and \quad \varepsilon_x = \frac{\sigma_x}{E}$$

$$\sigma_x = \frac{P}{A} = \frac{85\times 10^3}{\frac{\pi}{4}(30)^2} = 120.25\ MPa$$

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{120.25}{70\times 10^3} = 0.00172$$

$$\delta = \varepsilon_x . l = 0.00172(3\times 10^3) = 5.16\ mm$$

$$\Delta d = \varepsilon_y . d$$

$$\varepsilon_y = -v\frac{\sigma_x}{E} = -\frac{1}{3}\frac{(120.25)}{(70\times 10^3)} = -0.00057$$

$$\Delta d = -0.00057(30) = -0.0171\ mm$$

Change in volume (ΔV)=final volume – initial volume

$$\Delta V = \left[(30 - 0.0171)^2 \left(\frac{\pi}{4}\right)(3000 + 5.16)\right] - \left[(30)^2 \left(\frac{\pi}{4}\right)(3000)\right] = 1226.5\ mm^3$$

Example (6.6): Find a single force in x –direction that gives the same change in the direction parallel to x, for shown Figure (6.12). Take $v = {}^1/_3$ and $E = 70\ GPa$.

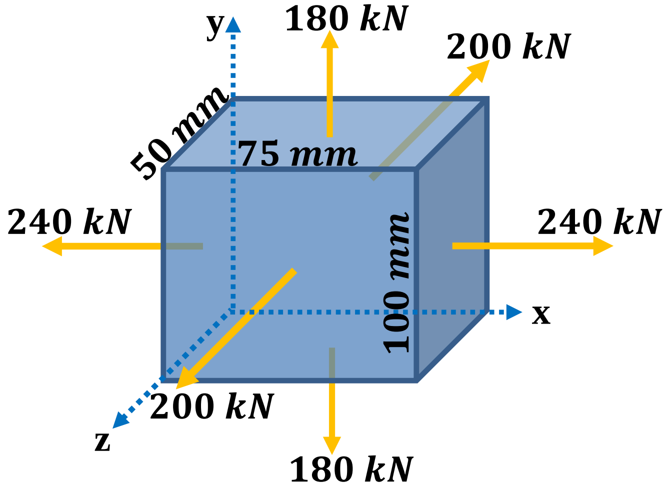


Figure (6.12)

Solution:

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\sigma_x = \frac{240 \times 10^3}{50(100)} = 48\ MPa$$

$$\sigma_y = \frac{180 \times 10^3}{50(75)} = 48\ MPa$$

$$\sigma_z = \frac{200 \times 10^3}{75(100)} = 26.67\ MPa$$

$$\varepsilon_x = \frac{48}{70 \times 10^3} - \frac{1}{3}\frac{48}{(70 \times 10^3)} - \frac{1}{3}\frac{26.67}{(70 \times 10^3)} = 3.302 \times 10^{-4}$$

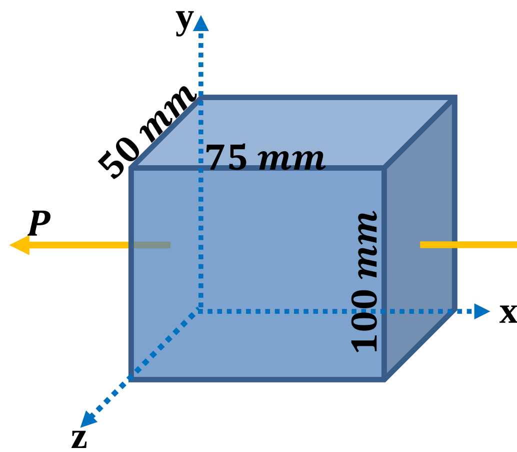
Uniaxial stress state is:

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$3.302 \times 10^{-4} = \frac{\frac{P}{50(100)}}{70 \times 10^3}$$

$$\rightarrow P = 115.57 \times 10^3 \ N$$

$$or \qquad P = 115.57 \ kN$$



Example (6.7): A uniform bar of length (***l***), cross – sectional area (***A***), and unit mass (***ρ***) is suspended vertically from one end as shown in Figure (6.13). Show that its total elongation is ($\delta = {}^{\rho g l^2}/_{2E}$). If the total mass of the bar is (***M***), show also that ($\delta = {}^{Mgl}/_{2EA}$).

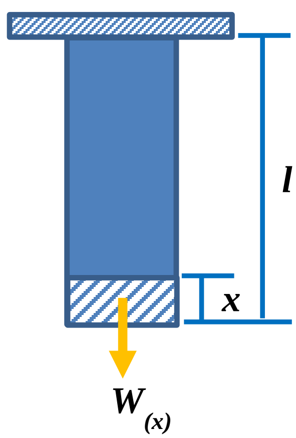


Figure (6.13)

Solution:

$$\rho_{(x)} = W_{(x)} = \gamma.Vol. = \rho(g)(A)(x)$$

$$\delta = \int_0^l \frac{\rho_{(x)}}{EA} dx = \int_0^l \frac{\rho(g)(A)}{EA} dx = \frac{\rho(g)}{E}\left[\frac{x^2}{2}\right]_0^l = \frac{\rho g l^2}{2E}$$

Total Mass= ***M*** , let total weight= ***M.g.l***

$$unit\ weight\ W_{(x)} = \frac{M.g}{l}(x)$$

$$\delta = \int_0^l \frac{\frac{M.g}{l}}{EA}(x)dx = \int_0^l \frac{M.g}{EA(l)}(x)dx$$

$$\delta = \frac{M.g}{EA(l)}\left[\frac{x^2}{2}\right]_0^l = \frac{M.g(l)}{2EA}$$

Example (6.8): The rigid bars shown in Figure (6.14) are separated by a roller at point (C) and pinned at point (A) and (D). A steel rod at point (B) helps support the load of (50 kN). Compute the vertical displacement of the roller at point (C).

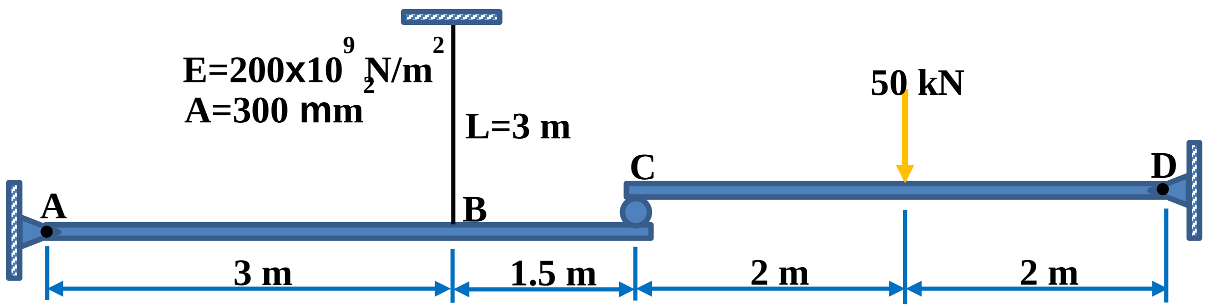
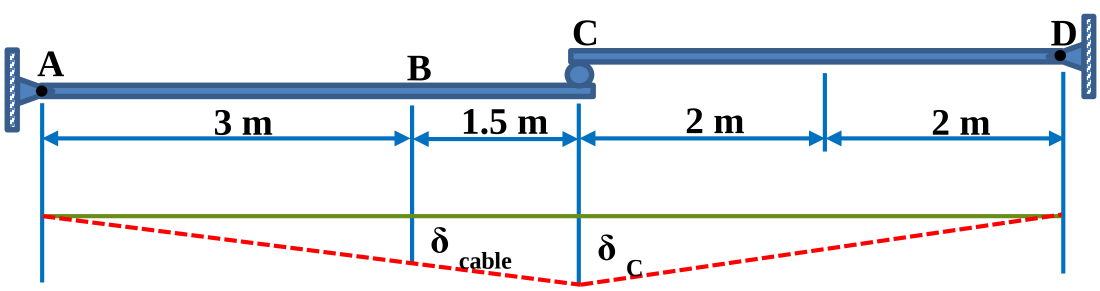


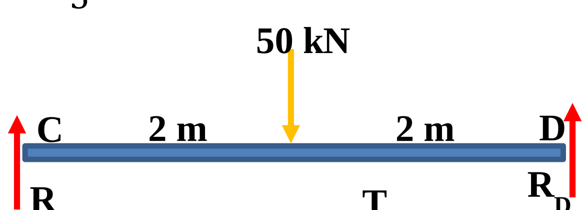
Figure (6.14)

Solution:



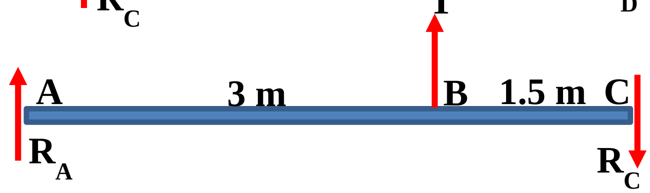
$$\frac{\delta_C}{4.5} = \frac{\delta_{Cable}}{3} \quad \rightarrow \delta_C = \frac{4.5}{3}\delta_{Cable}$$

Bar CD as F.B.D:



$$\circlearrowright \sum M_D = 0 \rightarrow \; R_C(4) = 50(2) \rightarrow R_C = 25\; kN \uparrow$$

Bar ABC as F.B.D:



$$\circlearrowright \sum M_A = 0 \rightarrow \; R_C(4.5) = T(3) \rightarrow T = 37.5\; kN \uparrow$$

$$\delta_{Cable}\frac{Pl}{EA} = \frac{37.5 \times 10^3}{200 \times 10^3(300)} = 1.875\,mm$$

$$\delta_C = \frac{4.5}{3}\delta_{Cable} = \frac{4.5}{3}(1.875) = 2.813\,mm$$

Example (6.9): A rod is composed of three segments and carries the axial loads as shown in Figure (6.15). Determine the stress in each material if the walls are rigid.

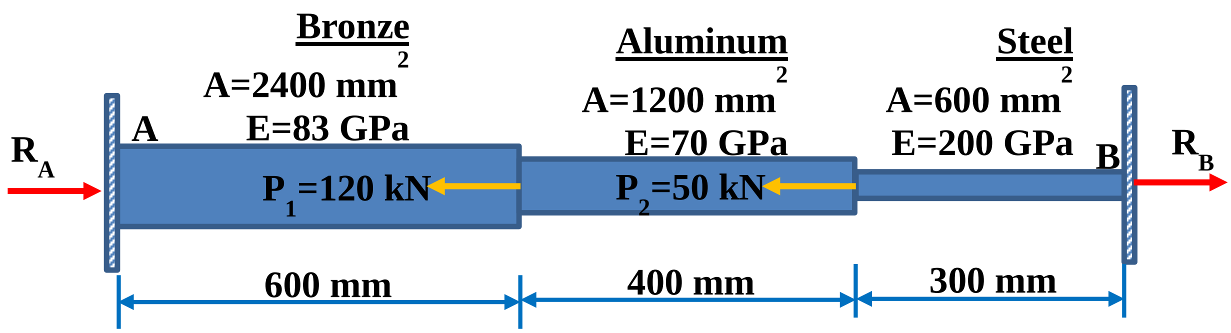
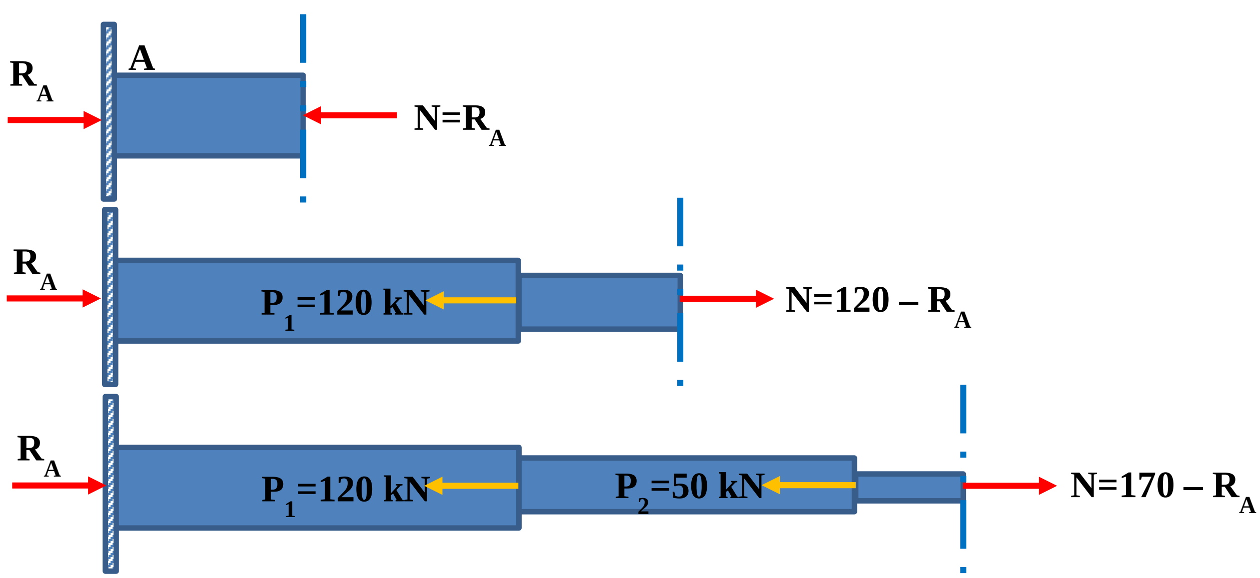


Figure (6.15)

Solution:



From equilibrium:

$$R_A + R_B = P_1 + P_2 \quad \rightarrow R_A + R_B = 170 \quad \dots\dots\dots.\dots(1)$$

From compatibility:

$$\boxed{\delta_{Br.} + \delta_{Al.} + \delta_{St.} = 0}$$

$$\frac{R_A \times 10^3(600)}{83 \times 10^3(2400)} + \frac{(120 - R_A) \times 10^3(400)}{70 \times 10^3(1200)} + \frac{(170 - R_A) \times 10^3(300)}{83 \times 10^3(600)} = 0$$

$$\rightarrow R_A = 96.99 \cong 97\ kN$$

Sub. in Equ.(1),get:

$$R_A + R_B = 170\ \rightarrow R_B = 170 - 97 = 73\ kN$$

$$\sigma_{Br.} = \frac{P}{A} = \frac{97 \times 10^3}{2400} = 40.42\ MPa$$

$$\sigma_{Al.} = \frac{P}{A} = \frac{(120 - 97) \times 10^3}{1200} = 19.17\ MPa$$

$$\sigma_{St.} = \frac{P}{A} = \frac{73 \times 10^3}{600} = 121.67\ MPa$$

Thermal Deformation

The strain due to an increase or decrease in temperature is:

$$\boldsymbol{\varepsilon = \alpha \Delta T}$$

Where $\alpha =$ is a coefficient of linear expansion, and

$\Delta T =$ Change in temperature.

- **ΔT is positive if increase in temperature and negative if decrease in temperature.**

The deformation due to this change in temperature is:

$$\boldsymbol{\delta l = \varepsilon(l) = \alpha \Delta T l}$$

Then the stress due to this change in temperature is:

$$\boldsymbol{\sigma = \varepsilon(E) = \alpha \Delta T E}$$

Example (6.10): A steel rod with a cross – sectional area of (150 mm^2) is stretched between two fixed points. The tensile load at (20C^o) is (5000N). What will be the stress at (-20C^o)? At what temperature will the stress be zero? Assume ($\alpha = 11.7\ \mu m/m.C^o$) and (E=200 GPa).

Solution:

$$\delta_P = \delta_T \rightarrow \frac{Pl}{EA} = \alpha \Delta T l$$

$$\frac{5000 - P}{200 \times 10^9 (150 \times 10^{-6})} = 11.7 \times 10^{-6} (-20 - 20)$$

$$\rightarrow P = 19040\ N$$

$$\sigma = \frac{P}{A} = \frac{19040}{150} = 127\ MPa$$

$$\frac{5000 - 0}{200 \times 10^9 (150 \times 10^{-6})} = 11.7 \times 10^{-6} (-20 - T)$$

$$\rightarrow T = -34.2\ C^o$$

Example (6.11): At (20 C°)a rigid slab having a mass of (55 Mg) is placed on two bronze rods and one steel rod as shown in Figure(6.16). At what temperature will the stress in steel rod be zero? For the steel (A=6000 mm^2, E=200 GPa, and $\alpha = 11.7\ \mu m/m.C^o$), and for bronze (A=6000 mm^2, E=83GPa, and $\alpha = 19\ \mu m/m.C^o$).

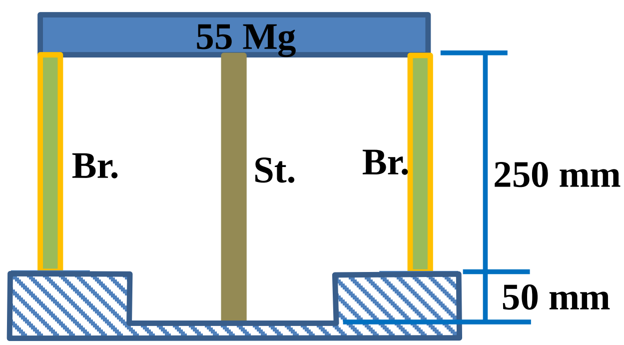
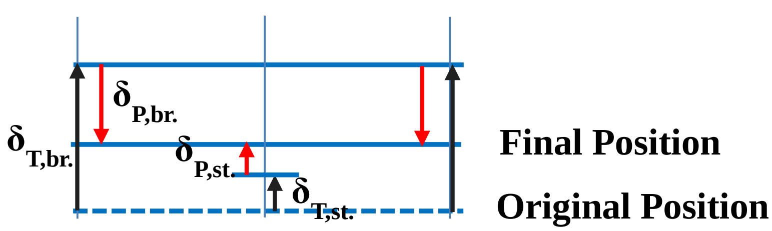


Figure (6.16)

Solution:



From compatibility:

$$\delta_{T,st} + \delta_{P,st} = \delta_{T,br} - \delta_{P,br}$$

$$\alpha \Delta T l_{st} + \frac{P_{st} l_{st}}{(EA)_{st}} = \alpha \Delta T l_{br} - \frac{P_{br} l_{br}}{(EA)_{br}} \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

From equilibrium:

$$2P_{br} + P_{st} = W$$

Note that: $W = 55 \times 10^3(9.81) = 539550\ N$

$$2P_{br} + P_{st} = 539550$$

❖ Wanted the temperature when stress in steel be zero, this led to:

$$\sigma_{st} = 0\ \rightarrow P_{st} = 0\ ,\ then$$

$$2P_{br} + 0 = 539550\ \rightarrow P_{br} = 269770\ N$$

Sub. These values in Equ.(1), get:

$$11.7 \times 10^{-6}(T - 20)(300) + 0 = 19 \times 10^{-6}(T - 20)(250) - \frac{269770(250)}{83 \times 10^3(6000)}$$

$$3.51 \times 10^{-3}(T - 20) = 4.75 \times 10^{-3}(T - 20) - 0.1354$$

$$1.24 \times 10^{-3}(T - 20) = 0.1354$$

$$T = 129.22\ C^o$$

<u>Example (6.12):</u> For the assembly shown in Figure (6.17). Find:

a) Change in temperature so that the two bar just to be touched.

b) If change in temperature (100 C^o). Find stresses in each bar.

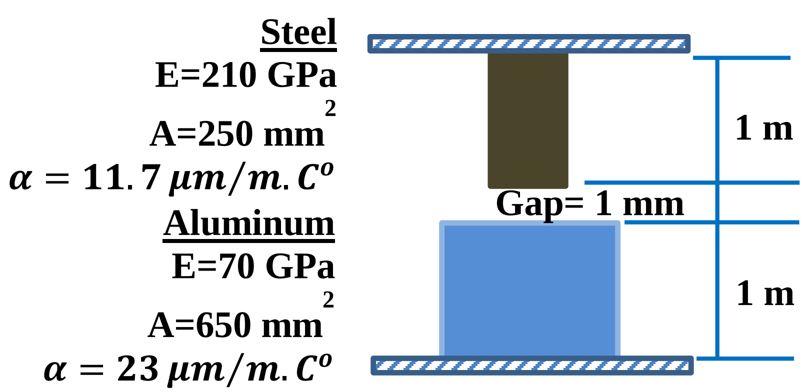


Figure (6.17)

<u>Solution:</u>

a)

$$\delta_{T,s} + \delta_{T,al} = 1$$

$$\alpha \Delta T l_s + \alpha \Delta T l_{al} = 1$$

$$11.7 \times 10^{-6} \Delta T (1 \times 10^3) + 23 \times 10^{-6} \Delta T (1 \times 10^3) = 1$$

$$0.0117 \Delta T + 0.023 \Delta T = 1$$

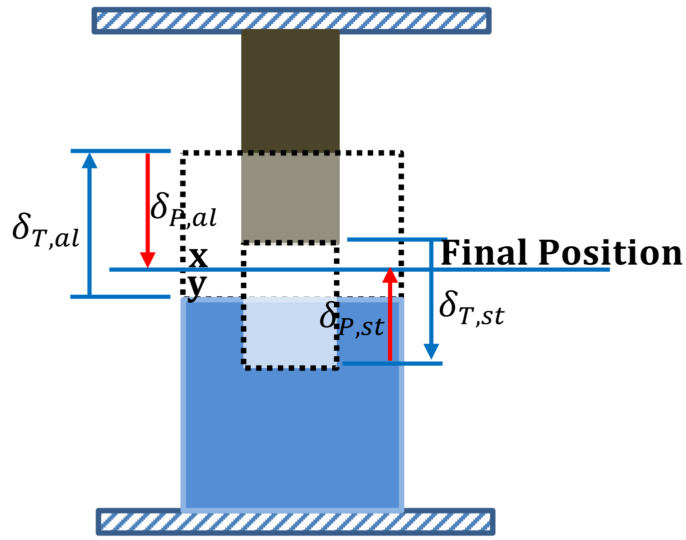
$$\Delta T = 28.82\ C^o$$

b)

$$x + y = 1 \quad \ldots\ldots\ldots.\ldots\ldots.(1)$$

$$x = \delta_{T,s} - \delta_{P,s} \quad \ldots\ldots\ldots\ldots(2)$$

$$y = \delta_{T,a} - \delta_{P,a} \quad \ldots\ldots\ldots\ldots(3)$$



Sub. Equ.(2) and (3) into Equ.(1), get:

$$(\delta_{T,s} - \delta_{P,s}) + (\delta_{T,a} - \delta_{P,a}) = 1$$

$$\left[11.7 \times 10^{-6}(100)(1 \times 10^3) - \frac{P_s(1 \times 10^3)}{210 \times 10^3(250)}\right] + \left[23 \times 10^{-6}(100)(1 \times 10^3) - \frac{P_a(1 \times 10^3)}{70 \times 10^3(650)}\right] = 1$$

$$1.17 - 1.905 \times 10^{-5} P_s + 2.3 - 2.198 \times 10^{-5} P_a = 1$$

From equilibrium:

$$P_a = P_s$$

$$1.17 - 1.905 \times 10^{-5} P_s + 2.3 - 2.198 \times 10^{-5} P_s = 1$$

$$\rightarrow\ P_s = P_a = 60199.85\ N$$

$$\rightarrow\ \sigma_s = \frac{60199.85}{250} = 240.8\ MPa$$

$$and\ \ \sigma_a = \frac{60199.85}{650} = 92.62\ MPa$$

Problems

Problem 6.1: The 4-mm-diameter cable *BC* is made of a steel with *E*=200 GPa. Knowing that the maximum stress in the cable must not exceed 190 MPa and that the elongation of the cable must not exceed 6 mm, find the maximum load **P** that can be applied as shown in Figure (6.18).

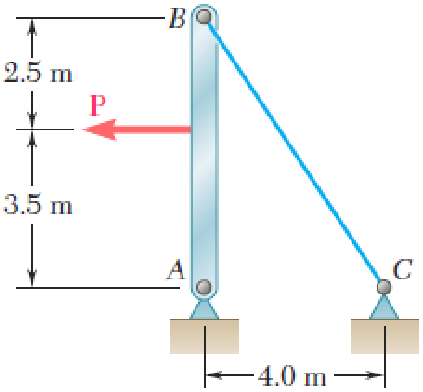


Figure (6.18)

Problem 6.2: The aluminum rod *ABC* (E=10.1x10^6 psi) (Figure 6.19), which consists of two cylindrical portions *AB* and *BC*, is to be replaced with a cylindrical steel rod *DE* (E=29x10^6 psi) of the same overall length. Determine the minimum required diameter ***d*** of the steel rod if its vertical deformation is not to exceed the deformation of the aluminum rod under the same load and if the allowable stress in the steel rod is not to exceed 24 ksi.

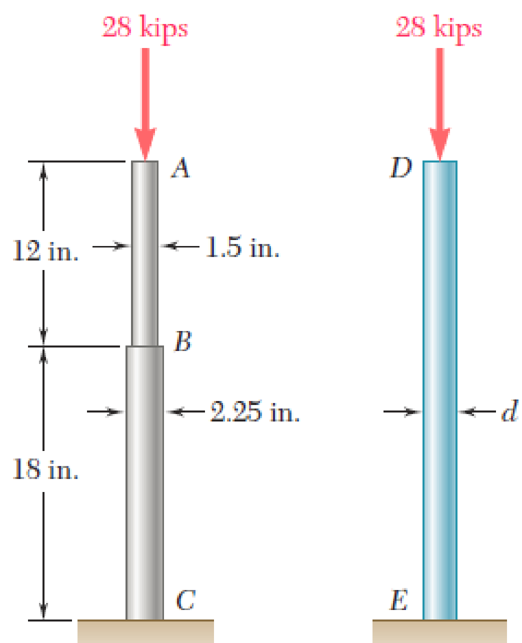


Figure 6.19

Problem 6.3: A rigid, weightless beam *BD* supports a load *P* and is, in turn, supported by two hanger rods, (1) and (2), as shown in Figure (6.20). The rods are initially the same length *L*=6 ft and are made of the same material. Their rectangular cross sections have original dimensions (w_1=1.5 in., t_1=0.75 in.) and (w_2=2.0 in., t_2=1.0 in.), respectively. (a) At what location, ***b***, must the load ***P*** act if the axial stress in the two bars is to be the same, i.e.,$\sigma_1 = \sigma_2$? (b) What is the magnitude of this tensile stress if a load of ***P*=40** kips is applied at the location determined in Part (a)?

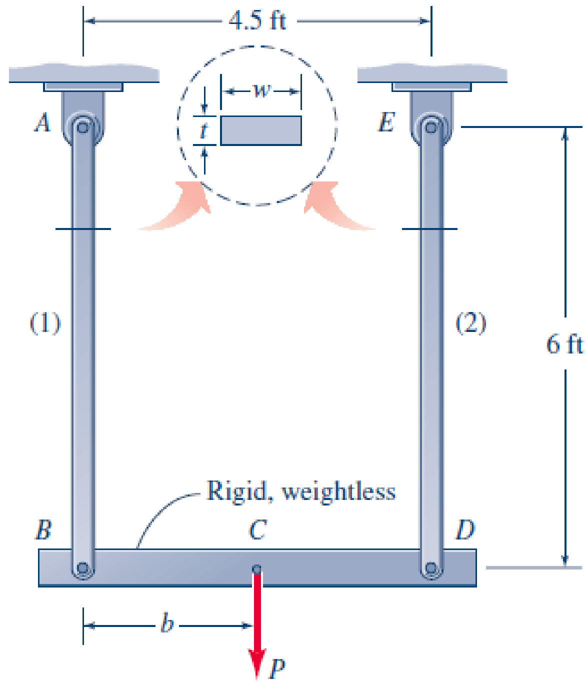


Figure (6.20)

Problem 6.4: Each member of the truss in Figure (6.21) is a solid circular rod with diameter d=10 mm. Determine the axial stress σ_1 in the truss member (1) and the axial stress σ_6 in the truss member (6).

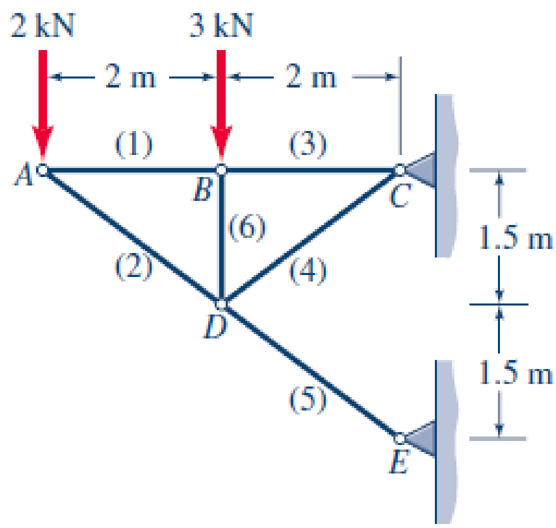


Figure (6.21)

Problem 6.5: The data in Table (6.1) was obtained in a tensile test of a flat-bar steel specimen having the dimensions shown in Figure (6.22).

(a) Plot a curve of engineering stress, σ , versus engineering strain, ϵ , using the given data.

(b) Determine the modulus of elasticity of this material.

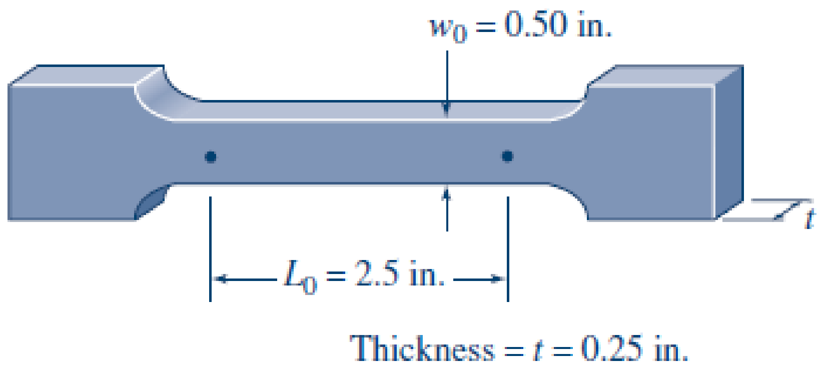


Figure (6.22)

Table (6.1): Tension-test Data; Flat Steel Bar

P **(kips)**	**Δ*L*** **(in.)**	***P*** **(kips)**	**Δ*L*** **(in.)**
1.2	0.0008	6.25	0.0060
2.4	0.0016	6.50	0.0075
3.6	0.0024	6.65	0.0100
4.8	0.0032	6.85	0.0125
5.7	0.0040	6.90	0.0150
5.95	0.0050	—	—

Problem 6.6: Under a compressive load of P=24 kips, the length of the concrete cylinder in Figure (6.23) is reduced from 12 in. to 11.9970 in., and the diameter is increased from 6 in. to 6.0003 in. Determine the value of the modulus of elasticity, ***E***, and the value of Poisson's ratio, *v*. Assume linearly elastic deformation.

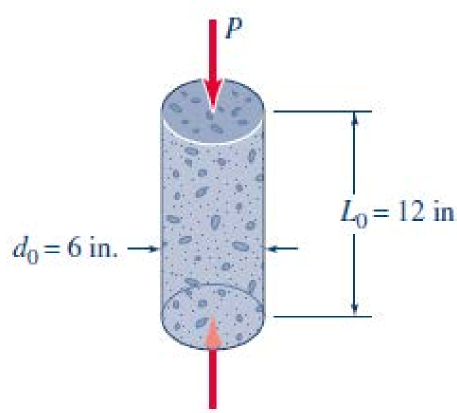


Figure (6.23)

Problem 6.7: An angle bracket, whose thickness is ***t*=12.7 mm**, is attached to the flange of a column by two 15-mm-diameter bolts, as shown in Figure (6.24). A floor joist that frames into the column exerts a uniform downward pressure of ***p*=2** MPa on the top face of the angle bracket. The dimensions of the loaded face are ***L*=152 mm** and ***b*=76 mm**. Determine the average shear stress, $\tau_{av.}$, in the bolts. (Neglect the friction between the angle bracket and the column.)

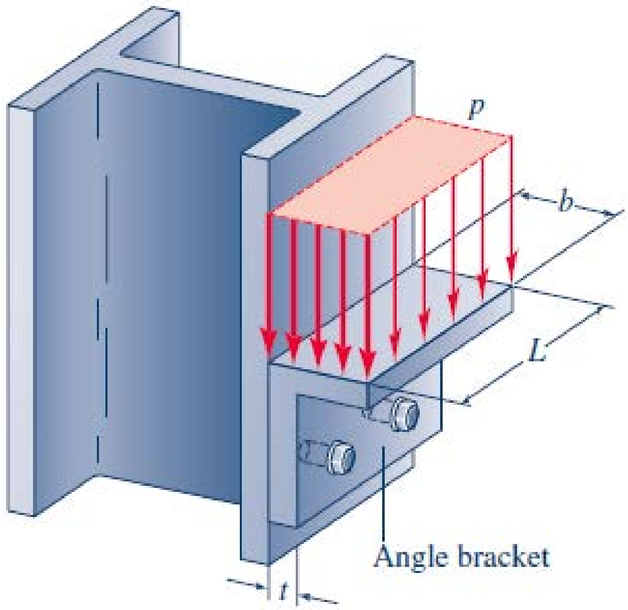


Figure (6.24)

Problem 6.7: A thin, rectangular plate shown in Figure (6.25) is subjected to a uniform biaxial state of stress (σ_x, σ_y). All other components of stress are zero. The initial dimensions of the plate are L_x=4 in. and L_y=2 in., but after the loading is applied, the dimensions are $L_1^* = 4.00176\ in.$, and $L_2^* = 2.00344\ in.$ If it is known that $\sigma_x = 10\ ksi$ and E=10x10^3 ksi, (a) what is the value of Poisson's ratio? (b) What is the value of σ_y?

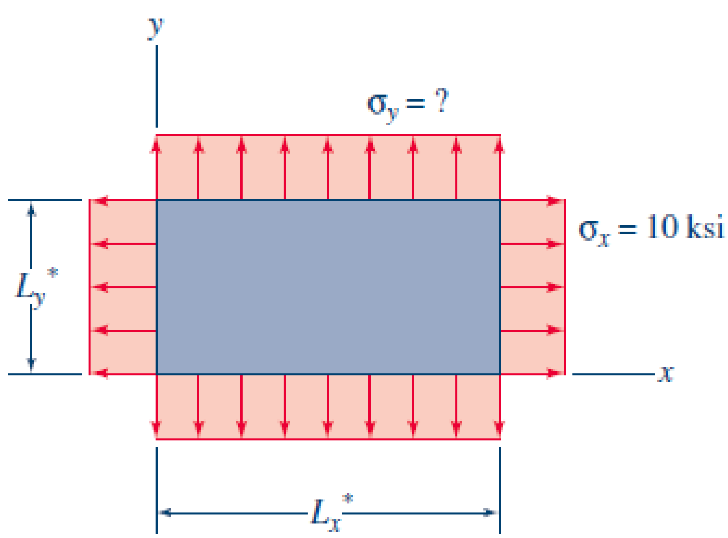


Figure (6.25)