Real Numbers

Types of Real Numbers $\mathbb{R}$

1. **Natural numbers** $\mathbb{N} = \{1,2,3,4,\cdots\}$

2. **Integer numbers** $\mathbb{Z} = \{0, \mp 1, \mp 2, \mp 3, \mp 4, \cdots\}$

3. **Rational numbers** $\mathbb{Q}$

$$r = \frac{m}{n} \quad : \quad m \in Z \,, n \in N$$

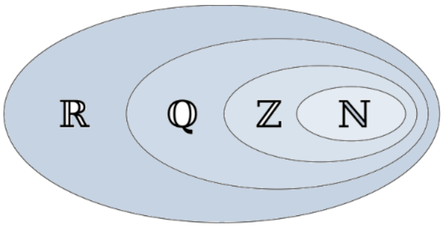
Examples

$$\frac{1}{2}, \quad \frac{5}{3}, \quad -\frac{16}{5}, \quad 4 = \frac{16}{4}, \quad 0.2 = \frac{2}{10}$$

4. **Irrational Numbers**: These are numbers that cannot be expressed as a ratio of integers.

Examples

$$\sqrt{2} \;, \sqrt[3]{5} \,, \pi \,, 2^{\sqrt{3}} \;, \; e$$

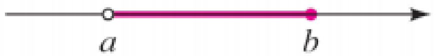


Intervals

Certain sets of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the open interval from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). The closed interval from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{x : a < x < b\} \quad , \quad [a, b] = \{x : a \le x \le b\}$$

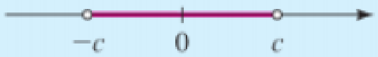
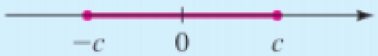
The following table lists the possible types of intervals

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	
$[a, b)$	$\{x \mid a \leq x < b\}$	
$(a, b]$	$\{x \mid a < x \leq b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \leq x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Absolute Value: The absolute value of a number a, denoted by $|a|$.

$$|a| = \begin{cases} a & if \quad a \geq 0 \\ -a & if \quad a < 0 \end{cases}$$

Properties of Absolute Value Inequalities

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \leq c$	$-c \leq x \leq c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \geq c$	$x \leq -c$ or $c \leq x$	

Inequalities:

An inequality is an expression involving one of the symbols $\geq, \leq, >$ or $<$

When we are asked to solve an inequality, the inequality will contain an unknown variable, say x. Solving means obtaining all values of x for which the inequality is true.

Properties:

1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality sign unchanged.

$$(\text{If } x < a \text{ then } x \mp b < a \mp b.)$$

2. Multiplying or dividing both sides by a positive number leaves the inequality sign unchanged.

$$\left(\text{If } x < a \text{ and } b > 0 \text{ then } xb < ab \text{ and } \frac{x}{b} < \frac{a}{b}.\right)$$

3. Multiplying or dividing both sides by a negative number reverses the inequality.

$$\left(\text{If } x < a \text{ and } b < 0 \text{ then } xb > ab \text{ and } \frac{x}{b} > \frac{a}{b}.\right)$$

Example 1: Solve $3x - 5 \leq 13$.

$$3x - 5 \leq 13 \Rightarrow 3x \leq 18 \Rightarrow x \leq 6$$

Example 2: Solve $-2x - 4 \leq -10$.

$$-2x - 4 \leq -10 \Rightarrow -2x \leq -6 \Rightarrow x \geq 3$$

Example 3: Solve $|2x - 5| \leq 7$.

$$|2x - 5| < 7 \Rightarrow -7 < 2x - 5 < 7$$
$$\Rightarrow -7 + 5 < 2x < 7 + 5$$
$$\Rightarrow -2 < 2x < 12$$
$$\Rightarrow -1 < x < 6$$

Example 4: Solve $|2 - 5x| \geq 3$.

$$|2 - 5x| \geq 3 \Rightarrow 2 - 5x \geq 3 \text{ or } 2 - 5x \leq -3$$
$$\Rightarrow -5x \geq 1 \text{ or } -5x \leq -5$$
$$\Rightarrow x \leq -\frac{1}{5} \text{ or } x \geq 1$$

The solution is

$$\left\{x: x \leq -\frac{1}{5}\right\} \cup \{x: x \geq 1\}$$

Solve quadratic inequality:

A quadratic inequality can be written in one of the standard forms

$ax^2 + bx + c < 0 \quad ax^2 + bx + c > 0 \quad ax^2 + bx + c \le 0 \quad ax^2 + bx + c \ge 0$

where a , b and c are real numbers and $a \neq 0$.

To solve a quadratic inequality in one variable, we will use the following steps to find the values of the variable that make the inequality true.

1- Write the inequality in standard form and solve its related quadratic equation.

2- Locate the solutions (called critical numbers) of the related quadratic equation on a number- line.

3- Test each interval on the number line created in step 2 by choosing a test value from the interval and determining whether it satisfies the inequality. The solution set includes the intervals whose test value make the inequality true.

4- Determine whether the endpoints of the intervals are included in the solution set.

Example 5: Solve $x^2 + x < 6$.

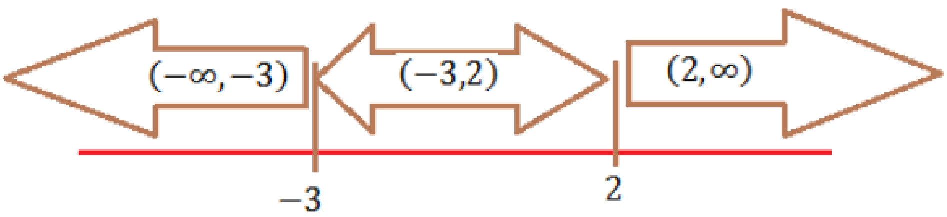
Solution: $x^2 + x < 6 \quad \Rightarrow \quad x^2 + x - 6 < 0$

We will solve the related quadratic equation $x^2 + x - 6 = 0$

$(x-2)(x+3) = 0 \quad \Rightarrow \quad x = 2 , -3$

These two critical numbers will separate the number-line into three intervals

$(-\infty, -3) \quad (-3,2) \quad (2, \infty)$



If we choose $-4 \in (-\infty, -3) \Rightarrow (-4)^2 + (-4) - 6 < 0 \Rightarrow 6 < 0$ False ×

If we choose $0 \in (-3,2) \Rightarrow (0)^2 + 0 - 6 < 0 \Rightarrow -6 < 0$ True √

If we choose $3 \in (2, \infty) \Rightarrow (3)^2 + 3 - 6 < 0 \Rightarrow 6 < 0$ False ×

Then the solution is the interval $(-3,2)$

Example 6: Solve $15 + 2x - x^2 \leq 0.$

Solution: We will solve the related quadratic equation $15 + 2x - x^2 = 0.$

$$x^2 - 2x - 15 = 0 \Rightarrow (x-5)(x+3) = 0 \Rightarrow x = 5, -3$$

These two critical numbers will separate the number-line into three intervals

$$(-\infty, -3], [-3,5] \text{ and } [5, \infty)$$

If we choose $-4 \in (-\infty, -3] \Rightarrow 15 + 2 \times (-4) - (-4)^2 \leq 0 \Rightarrow -9 \leq 0$ True √

If we choose $0 \in [-3,5] \Rightarrow 15 + 2 \times 0 - (0)^2 \leq 0 \Rightarrow 15 \leq 0$ False ×

If we choose $6 \in [5, \infty) \Rightarrow 15 + 2 \times 6 - (6)^2 \leq 0 \Rightarrow -9 \leq 0$ True √

Then the solution is $(-\infty, -3] \cup [5, \infty)$

Example 7: Solve $\dfrac{x-4}{x+1} \geq 0$

Solution:

The zero of numerator is $x - 4 = 0 \Rightarrow x = 4$

The zero of denominator is $x + 1 = 0 \Rightarrow x = -1$

These two critical numbers will separate the number-line into three intervals

$$(-\infty, -1), \quad (-1, 4] \text{ and } [4, \infty)$$

If we choose $-2 \in (-\infty, -1) \Rightarrow \dfrac{x-4}{x+1} \geq 0 \Rightarrow \dfrac{-2-4}{-2+1} \geq 0 \Rightarrow 6 \geq 0$ True √

If we choose $0 \in (-1, 4] \Rightarrow \dfrac{x-4}{x+1} \geq 0 \Rightarrow \dfrac{0-4}{0+1} \geq 0 \Rightarrow -4 \geq 0$ False ×

If we choose $5 \in [4, \infty) \Rightarrow \dfrac{x-4}{x+1} \geq 0 \Rightarrow \dfrac{5-4}{5+1} \geq 0 \Rightarrow \dfrac{1}{6} \geq 0$ True √

Then the solution is $(-\infty, -1) \cup [4, \infty)$

Exercises

Solve the inequalities

1. $2x^2 + 4x \leq x^2 + 32$
2. $|2x - 5| < 9$
3. $3x^2 + 2x \geq 2x^2 + 8$
4. $|x^2 - 5x| < 6$
5. $\dfrac{3x+9}{3-x} \geq 0$
6. $\dfrac{6-3x}{5-x} \leq 0$