Ministry of Higher Education and Scientific Research

Al-Mustaqbal University College

Department of Technical Computer Engineering

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Mathematics II

2nd Stage

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1. Sequences

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \dots . a_{\mathrm{n}}$$

The number a_1 is called the *first term,* a_2 is the *second term,* and in general a_{n} is the *nth term.* We will deal exclusively with infinite sequences and so each term will have a successor $a_{\mathrm{n}+1}$.

Notice that for every positive integer n there is a corresponding number a_{n} and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_{n} instead of the function notation $f(n)$ for the value of the function at the number.

Definition:

The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

i. Convergence and divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$$

whose terms approach 0 as n gets large, and in the sequence.

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, 1 - \frac{1}{n}, \dots\right\}$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$

Definition:

The sequence $\{a_n\}$ converges to the number L if for every positive number P there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

ii. Limit properties of the sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

2. Infinite series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an infinite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Definition:

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Properties of series aljebra

If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

i. Integral test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(x)$, where f is continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). If:

$$\int_N^{\infty} f(x)\ dx \quad \text{is convergant then the} \sum_{n=N}^{\infty} a_n \text{ is also convergant}$$

ii. The Ratio Test

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio a_{n+1}/a_n. For a geometric series $\sum ar^n$, this rate is a constant and the $ar^{n+1}/ar^n = r$. series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

Let $\sum a_n$ be any series and suppose that:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then

1. The series converges absolutely if $\rho < 1$.
2. The series diverges if $\rho > 1$ or ρ is infinite.
3. The test is inconclusive if $\rho = 1$.

iii. Geometric series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

The sum of geometric series is defined as:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

Definition:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

iv. Power series

We begin with the formal definition, which specifies the notation and terminology used for
power series.

Definition:

A power series about x = 0 is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots . \quad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Radius of convergence of power series

A power series $\sum^{\infty} c_k x^k$ will converge only for certain values of x. For instance, $\sum_{k=0}^{\infty} x^k$ converges for $-1 < x < 1$. In general, there is always an interval (-R, R) in which a power series converges, and the number R is called the radius of convergence (while the interval itself is called the interval of convergence). The quantity R is called the radius of convergence because, in the case of a power series with complex coefficients, the values of x with $|x| < R$ form an open disk with radius R.

To find the radius of convergance we follow the steps:

1. Use the ratio test and evaluate the limit and but the ratio < 1

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

2. Solve the inequality to find the interval of x.
3. Test the endpoint value of the interval for convergance.

Examples

1. Sequences

i. Find the first four terms of the following sequences:

a- $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ b- $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ c- $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$

Solution

a- $a_n = \frac{n}{n+1}$ $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

b- $a_n = \sqrt{n-3},\ n \geqslant 3$ $\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$

c- $a_n = \cos\frac{n\pi}{6},\ n \geqslant 0$ $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$

ii. Determine whether the sequence converges or diverges. If it converges, find the limit

a- $a_n = 2 + (0.1)^n$ b- $a_n = \frac{n + (-1)^n}{n}$ c- $a_n = \frac{1-2n}{1+2n}$

d- $a_n = \frac{1-5n^4}{n^4+8n^3}$ e- $a_n = \frac{n+3}{n^2+5n+6}$ f- $a_n = \frac{n^2-2n+1}{n-1}$

g- $a_n = \frac{1-n^3}{70-4n^2}$

Solution

a- $\lim_{n\to\infty} 2 + (0.1)^n = 2 \Rightarrow$ converges

b- $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1 \Rightarrow$ converges

c- $\lim_{n\to\infty} \frac{1-2n}{1+2n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)-2}{\left(\frac{1}{n}\right)+2} = \lim_{n\to\infty} \frac{-2}{2} = -1 \Rightarrow$ converges

d- $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^4}\right)-5}{1+\left(\frac{8}{n}\right)} = -5 \Rightarrow$ converges

e- $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim_{n\to\infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n\to\infty} \frac{1}{n+2} = 0 \Rightarrow$ converges f- $\lim_{n \to \infty} \frac{n^2 - 2n + 1}{n - 1} = \lim_{n \to \infty} \frac{(n-1)(n-1)}{n-1} = \lim_{n \to \infty} (n - 1) = \infty \Rightarrow$ diverges

g- $\lim_{n \to \infty} \frac{1 - n^3}{70 - 4n^2} = \lim_{n \to \infty} \frac{\left(\frac{1}{n^2}\right) - n}{\left(\frac{70}{n^2}\right) - 4} = \infty \Rightarrow$ diverges

2. <u>Infinite series</u>

i. <u>Integral test</u>

Use the Integral Test to determine if the following series is convergent or divergent.

a- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b- $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$ c- $\sum_{n=1}^{\infty} \frac{1}{n + 4}$ d- $\sum_{n=1}^{\infty} e^{-2n}$

e- $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$

Solution

a- $f(x) = \frac{1}{x^2}$ is positive, continuous, and decreasing for $x \geq 1$;

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b \to \infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_1^{\infty} \frac{1}{x^2}\, dx \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

b- $f(x) = \frac{1}{x^2+4}$ is positive, continuous, and decreasing for $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2+4}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^2+4}\, dx = \lim_{b \to \infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^b$$

$$= \lim_{b \to \infty} \left(\frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}\frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{1}{2}$$

$$\int_1^{\infty} \frac{1}{x^2+4}\, dx \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+4} \text{ converges}$$

c- $f(x) = \frac{1}{x+4}$ is positive, continuous, and decreasing for $x \geq 1$:

$$\int_1^\infty \frac{1}{x+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+4}\,dx = \lim_{b\to\infty} \Big[\ln|x+4|\Big]_1^b$$

$$= \lim_{b\to\infty} (\ln|b+4| - \ln 5) = \infty$$

$$\Rightarrow \int_1^\infty \frac{1}{x+4}\,dx \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+4} \text{ diverges}$$

d- $f(x) = e^{-2x}$ is positive, continuous, and decreasing for $x \geq 1$:

$$\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty} \int_1^b e^{-2x}\,dx = \lim_{b\to\infty} \Big[-\tfrac{1}{2}e^{-2x}\Big]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{2e^{2b}} + \frac{1}{2e^2}\right) = \frac{1}{2e^2}$$

e- $f(x) = \frac{x}{x^2+4}$ is positive and continuous for $x \geq 1$, $f'(x)$

$f'(x) = \frac{4-x^2}{(x^2+4)^2} < 0$ for $x > 2$, thus f is decreasing for $x \geq 3$;

$$\int_3^\infty \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \int_3^b \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \Big[\tfrac{1}{2}\ln(x^2+4)\Big]_3^b$$

$$= \lim_{b\to\infty} \left(\tfrac{1}{2}\ln(b^2+4) - \tfrac{1}{2}\ln(13)\right) = \infty$$

$$\Rightarrow \int_3^\infty \frac{x}{x^2+4}\,dx \quad \text{diverges} \Rightarrow \sum_{n=3}^{\infty} \frac{n}{n^2+4} \text{diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4} = \frac{1}{5} + \frac{2}{8} + \sum_{n=3}^{\infty} \frac{n}{n^2+4} \text{ diverges}$$

ii. **<u>Ratio test</u>**

Use the Ratio Test to determine if each series converges absolutely or diverges

a. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$ b. $\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$ c. $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$ d. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

Solution

$\frac{2^{n+1}}{n\cdot 3^{n-1}} > 0$ for all $n \geq 1$

a-

$$\lim_{n\to\infty}\left(\frac{\frac{2^{(n+1)+1}}{(n+1)\cdot 3^{(n+1)-1}}}{\frac{2^{n+1}}{n\cdot 3^{n-1}}}\right) = \lim_{n\to\infty}\left(\frac{2^{n+1}\cdot 2}{(n+1)\cdot 3^{n-1}\cdot 3}\cdot\frac{n\cdot 3^{n-1}}{2^{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n}{3n+3}\right) = \lim_{n\to\infty}\left(\frac{2}{3}\right) = \frac{2}{3} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{2^{n+1}}{n\cdot 3^{n-1}} \text{ converges}$$

b- $\frac{n^4}{4^n} > 0$ for all $n \geq 1$

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^4}{4^n\cdot 4}\cdot\frac{4^n}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^4+4n^3+6n^2+4n+1}{4n^4}\right) = \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{n}+\frac{3}{2n^2}+\frac{1}{n^3}+\frac{1}{4n^4}\right) = \frac{1}{4} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n^4}{4^n} \text{ converges}$$

c- $\frac{n+2}{3^n} > 0$ for all $n \geq 1$; $\lim_{n\to\infty}\left(\frac{\frac{(n+1)+2}{3^{n+1}}}{\frac{n+2}{3^n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{n+3}{3^n\cdot 3}\cdot\frac{3^n}{n+2}\right) = \lim_{n\to\infty}\left(\frac{n+3}{3n+6}\right) = \lim_{n\to\infty}\left(\frac{1}{3}\right) = \frac{1}{3} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n+2}{3^n} \text{ converges}$$

d- $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^3\,2^{n+1}}\cdot\frac{n^3 2^n}{3^n}$

$$= \lim_{n\to\infty}\frac{n^3}{(n+1)^3}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n^3 2^n} \quad \text{diverges}$$

3. <u>Geometric series</u>

Show if the following geometric series converge or diverge and find the sum if they converge:

a- $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ b- $\sum_{n=2}^{\infty} \frac{1}{4^n}$ c- $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

d- $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$ e- $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

Solution

a- $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ $\boldsymbol{r = \frac{-1}{4}}$ $\boldsymbol{a = 1}$

since $|\boldsymbol{r}| < \mathbf{1}$ $\Longrightarrow$ the sum of the series converges to :

$$\text{Sum} = \frac{\boldsymbol{a}}{\mathbf{1} - \boldsymbol{r}} = \frac{1}{1 - \left(-\frac{1}{4}\right)} = \frac{1}{1 + \left(\frac{1}{4}\right)} = \frac{4}{5}$$

b- $\sum_{n=2}^{\infty} \frac{1}{4^n}$ $\boldsymbol{r = \frac{1}{4}}$ $\boldsymbol{a = 1}$

since $|\boldsymbol{r}| < \mathbf{1}$ $\Longrightarrow$ the sum of the series converges to :

$$\text{Sum} = \frac{\text{a}}{1 - \text{r}} = \frac{1}{1 + \frac{1}{4}} = \frac{4}{3} \quad \text{for n} = (0, \infty)$$

Since n =2 then

$$\sum_{n=2}^{\infty} \frac{1}{4^n} = \text{Sum} - \frac{1}{4^0} - \frac{1}{4^1} = \frac{1}{12}$$

c- $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$ $\boldsymbol{r = \frac{-1}{4}}$ $\boldsymbol{a = 5}$

since $|\boldsymbol{r}| < \mathbf{1}$ $\Longrightarrow$ the sum of the series converges to :

$$\text{Sum} = \frac{\boldsymbol{a}}{\mathbf{1} - \boldsymbol{r}} = \frac{5}{1 - \left(-\frac{1}{4}\right)} = 4$$

d- $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

$$r_1 = \frac{1}{2} \qquad a_1 = 5 \qquad , \qquad r_2 = \frac{1}{3} \qquad a_2 = 1$$

since $|r_1| < 1$ **and** $|r_2| < 1 \Longrightarrow$ the sum of the series converges to :

$$\text{Sum} = S_1 + S_2 = \frac{5}{1-\left(\frac{1}{2}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 10 - \frac{3}{2} = \frac{17}{2}$$

e- $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2^{n+1}}{5^n}\right) \qquad r = \frac{2}{5} \qquad a = 2$

since $|r| < 1 \Longrightarrow$ the sum of the series converges to :

$$\text{Sum} = \frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{10}{3}$$

4. <u>Power series</u>

Find the radius and interval of comvergance of the following power series:

a- $\displaystyle\sum_{n=0}^{\infty} x^n$ b- $\displaystyle\sum_{n=0}^{\infty}(x+5)^n$ c- $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$ d- $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$

e- $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{n^2\,3^n}$

Solution

a- Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$$

Test for endpoints

when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n$, a divergent

when $x = 1$ we have $\sum^{\infty} 1$, a divergent series

the radius is 1; the interval of convergence is $-1 < x < 1$

b- Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| < 1 \Rightarrow |x+5| < 1 \Rightarrow -6 < x < -4$$

Test for endpoints

$$\sum_{n=1}^{\infty}(-1)^n\text{, a divergent series}$$

when $x = -6$ we have

when $x = -4$ we have $\sum_{n=1}^{\infty} 1$, a divergent series

the radius is 1; the interval of convergence is $-6 < x < -4$

c- Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right| < 1 \Rightarrow \frac{|x-2|}{10} < 1$$

$$\Rightarrow |x-2| < 10 \Rightarrow -10 < x-2 < 10 \Rightarrow -8 < x < 12$$

Test for endpoints

when $x = -8$ we have $\sum_{n=1}^{\infty}(-1)^n$, a divergent series

when $x = 12$ we have $\sum_{n=1}^{\infty} 1$, a divergent series

the radius is 10; the interval of convergence is $-8 < x < 12$

d- Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x-1)^n}\right| < 1$$

$$\Rightarrow |x-1|\sqrt{\lim_{n\to\infty}\frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$$

Test for endpoints

when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series

when $x = 2$ we have $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$, a divergent series

the radius is 1; the interval of convergence is $0 \le x < 2$

e- Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}}\cdot\frac{n^2 3^n}{(x-1)^n}\right| < 1$$

$$\Rightarrow |x-1|\lim_{n\to\infty}\left(\frac{n^2}{3(n+1)^2}\right) = \frac{1}{3}|x-1| < 1 \Rightarrow -2 < x < 4$$

Test for endpoints

when $x = -2$ we have $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, an absolutely convergent series when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{(3)^n}{n^2 \, 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absolutely convergent series.

the radius is 3; the interval of convergence is $-2 \leq x \leq 4$