جامـــــعـة الـمـــــســتـقـبـل
AL MUSTAQBAL UNIVERSITY

قســـم الامـــــــن الـــــــــــسيبرانـــــــــــي

DEPARTMENT OF CYBER SECURITY

SUBJECT:

FINITE SETS AND COUNTING PRINCIPLE

CLASS:

FIRST

LECTURER:

ASST. LECT. MUSTAFA AMEER AWADH

LECTURE: (3)

Finite Sets and Counting Principle:

A set is said to be finite if it contains exactly m distinct elements, where m denotes some nonnegative integer.
Otherwise, a set is said to be infinite.

For example:

- The empty set ϕ and the set of letters of English alphabet are finite sets,
- The set of even positive integers, {2,4, 6,…..}, is infinite.

If a set A is finite, we let n(A) or #(A) denote the number of elements of A.

Example:

If A ={1,2,a,w} then

$n(A) = \#(A) = |A| = 4$

Lemma: If A and B are finite sets and disjoint Then A ∪ B is finite set and

$$n(A \cup B) = n(A) + n(B)$$

Theorem (Inclusion–Exclusion Principle): Suppose A and B are finite sets. Then A ∪ B and A ∩ B are finite and

$$|A \cup B| = |A| + |B| - |A \cap B|$$

That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting n(A ∩ B) (exclusion) since its elements were counted twice. We can apply this result to obtain a similar formula for three sets:

Corollary:
If A, B, C are finite sets then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example (1) :
A= {1,2,3}
B= {3,4}
C= {5,6}

$A \cup B \cup C = \{1,2,3,4,5,6\}$
$|A \cup B \cup C| = 6$

$|A| = 3$, $|B| = 2$, $|C| = 2$
$A \cap B = \{3\}$, $|A \cap B| = 1$
$A \cap C =$
$\{ \}$, $|A \cap C| = 0$
$B \cap C = \{ \}$, $A \cap B \cap C = \{ \}$, $|B \cap C| = 0$, $|A \cap B \cap C| = 0$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 3 + 2 + 2 - 1 - 0 - 0 + 0 = 6$

Example (2):
Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class and suppose there are 20 names on both lists. Find the number of students:
(a) only on list A
(b) only on list B
(c) on list $A \cup B$

Solution:

(a) List A has 30 names and
20 are on list B;
hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 30 + 35 - 20 = 45.$$

Example (3):

Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian:

65 study French (F).

45 study German (G).

42 study Russian (R).

20 study French & German $F \cap G$.

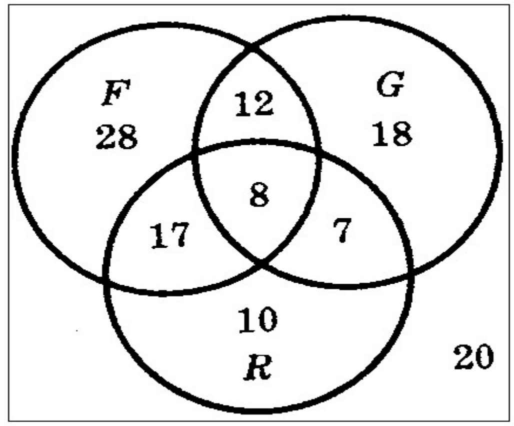
25 study French & Russian $F \cap R$.

15 study German & Russian $G \cap R$.

Find the number of students who study:

1) All three languages ($F \cap G \cap R$)

2) The number of students in each of the eight regions of the Venn diagram



Solution:

$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$
$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$
$100 = 92 + |F \cap G \cap R|$
$\therefore |F \cap G \cap R| = 8$ students study the 3 languages
$20 – 8 = 12$
$25 – 8 = 17$
$15 – 8 = 7$
$(F \cap G) - R$
$(F \cap R) - G$
$(G \cap R) – F$

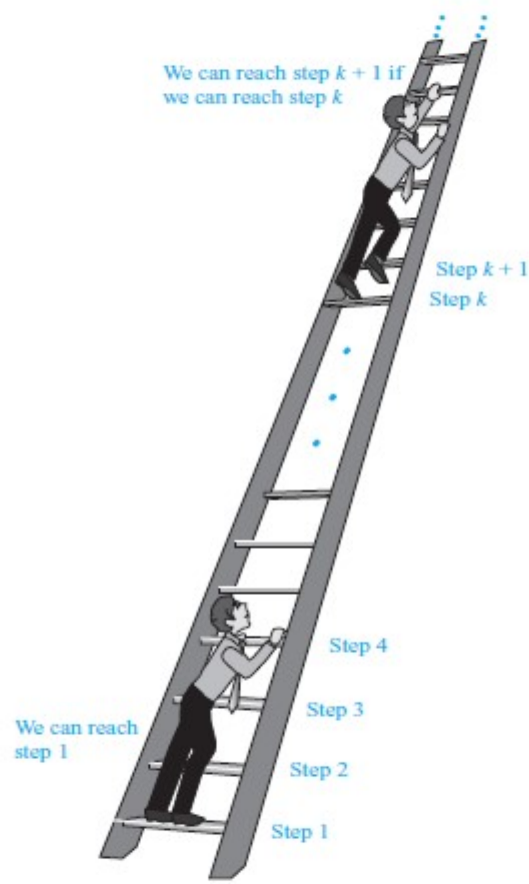
$65 – 12 – 8 – 17 = 28$ students study French only
$45 – 12 – 8 - 7 = 18$ students study German only
$42 – 17 – 8 - 7 = 10$ students study Russian only
$120 – 100 = 20$ students do not study any language

Mathematic induction:

Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder. We know two things:

l. We can reach the first rung of the ladder.

2. If we can reach a particular rung of the ladder, then we can reach the next rung. Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on. For example, after 100 uses of (2), we know that we can reach the 101 st rung.



We can verify using an important proof technique called mathematical induction. That is, we can show that *P(n)* is true for every positive integer *n,* where *P(n)* is the statement that we can reach the *nth* rung of the ladder.
Mathematical induction is an important proof technique that can be used to prove assertions of this type. Mathematical induction is used to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In general, mathematical induction can be used to prove statements that assert that *P(n)* is true for all positive integers *n,* where *P(n)* is a propositional function.

PRINCIPLE OF *MATHEMATICAL INDUCTION*
To prove that *P(n)* is true for all positive integers *n,* where *P(n)* is a propositional function, we complete two steps:

(i)BASIS *STEP:* We verify that *P(1)* is true.
(ii)INDUCTIVE STEP*:* We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers *k.*

EXAMPLE1:
Show that if *n* is a positive integer, then

$$1+2+ \ldots +n = \frac{n(n+1)}{2}$$

Prove P (for n ≥ 1)

Solution*:*

Let *P(n)* be the proposition that the sum of the first *n* positive integers is $n(n + 1)/2$

We must do two things to prove that *P(n)* is true for
$n = 1, 2, 3, ...$

Namely, we must show that *P(1)* is true and that the conditional statement *P(k)* implies *P(k* + 1) is true for $k = 1,2,3, ...$

(i)BASIS STEP: P(1) IS true, because $1 = \frac{1(1+1)}{2}$

left side =1 & Right side =2/2 = 1

left side = Right side

(ii)INDUCTIVE STEP: For the inductive hypothesis we assume that *P(k)* holds for an arbitrary positive integer *k.* That is, we assume that P(k) is true

$$1+2+ \dots +k = \frac{k(k+1)}{2}$$

Under this assumption, it must be shown that *P(k* + 1) is true, namely, that

to prove that P(k+1) is true

$$1 + 2 + 3 + 4 + \dots + k + (k+1) = 1/2 * k * (k+1) + (k+1)$$

$$= \frac{k (k+1) + 2 (k+ 1)}{2}$$

$$= \frac{(k+1) (k+ 2)}{2}$$

$$= 1/2\ (k + 1)(k + 2)$$

So P is true for all $n \geq k$

Example 2:

Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction. **Solution**:

The sums of the first *n* positive odd integers for $n = 1,2,3,4,5$ are:

$1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$,

$1+3+5+7=16$, $1 + 3 + 5 + 7 + 9 = 25$.

From these values it is reasonable to conjecture that the sum of the first *n* positive odd integers is n^2, that is,

$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let *P(n)* denote the proposition that the sum of the first *n* odd positive integers is n^2

(i)BASIS STEP: P(1) states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1.

(ii)INDUCTIVE STEP: we first assume the inductive hypothesis.

The inductive hypothesis is the statement that *P(k)* is true, that is,

$$1 + 3 + 5 + ... + (2k - 1) = k^2$$

(ii) n=k; Assuming P(k) is true,

We add (2k-1)+2 = 2K + 1 to both sides of P(k), obtaining:

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$
$$= (k + 1)^2$$

Which is P(k + 1).
That is, P(k + 1) is true whenever P(k) is true.
By the principle of mathematical induction:
P is true for all $n \geq k$.

Example 3:
Prove the following proposition (for $n \geq 0$):

$$P(n) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

solution:

(i) P(0) : left side =1

Right side $=2^1-1=1$

(ii) Assuming P(k) is true ; n=k

$$P(k) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$$

We add 2^{k+1} to both sides of P(k), obtaining

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is P(k +1). That is, P(k +1) is true whenever P(k) is true.
By the principle of induction:
P(n) is true for all n.