Subject Name: Calculus I1

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Lecture No. 3

Lecture Title: Functions.

CHAPTER THREE

FUNCTIONS

3.1 Functions: Functions are a tool for describing the real world in mathematical terms. An equation, a graph, a numerical table, or a verbal description can represent a function.

$$y = f(x) \text{ ("}y\text{ equals } f \text{ of } x\text{").}$$

DEFINITION: A function f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

3.2 Domain and Range

Domain: The set D of all possible input values is called the function's domain.

Range: The set of all values of f(x) as x varies throughout D is called the range of the function.

REMARKS:

1 – The symbol f represents the function, the letter x is the independent variable representing the input value of f, and y is the dependent variable or output value of f at x.

2 – The range may not include every element in the set Y.

3 – The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line.

4 – Often a function is given by a formula that describes how to calculate the output value from the input variable.

5 – When we define a function y = f(x) with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called natural domain.

6 – Changing the domain to which we apply a formula usually changes the range as well.

7 – When the range of a function is a set of real numbers, the function is said to be real- valued.

8 – The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals.

9 – The interva1s may be open, closed, or half open, and may be finite or infinite.

10 – The range of a function is not always easy to find.

11 – A function f is like a machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (Figure 1.1).

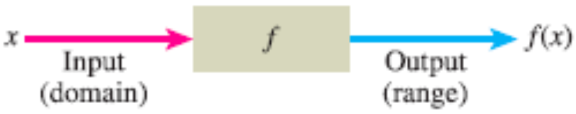


FIGURE 1.1 A diagram showing a function as a kind of machine.

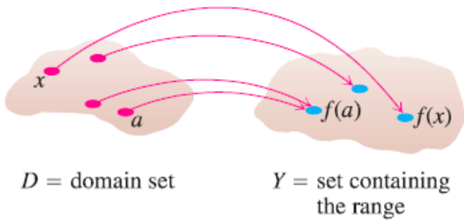


FIGURE 1.2 A function from a set *D* to a set *Y* assigns a unique element of *Y* to each element in *D*.

12 – _A function can have the same *value* at two different input elements in the domain, but each input element *x* is assigned a *single* output value *f(x)*.

EXAMPLE 1: Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of *x* for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution: 1 – _The formula $y = x^2$ gives a real y-value for any real number *x,* so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number *y* is the square of its own square root ,

$y = (\sqrt{y})^2$ for $y \geq 0$.

2 – _The formula $y = 1/x$ gives a real y-value for every *x* except $x = 0$. For consistency in the rules of arithmetic, we *cannot divide any number by zero.* The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers , is the set of all nonzero real numbers, since

$y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

3 – _The formula $y=\sqrt{x}$ gives a real y-value only if $x\geq 0$. The range of $y=\sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

4 – _In $y=\sqrt{4-x}$, the quantity $4-x$ cannot be negative . That is, $4-x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4-x}$ is $[0, \infty)$, the set of all nonnegative numbers.

5 – _The formula $y=\sqrt{1-x^2}$ gives a real *y-value* for every x in the closed interval from - 1 to 1. Outside this domain. $1-x^2$ is negative and its square root is not a real number. The values of $1-x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1-x^2}$ is [0, 1].

3.3 Graphs of Functions:

If *f* is a function with domain *D,* its **graph** consists of the points in the Cartesian plane whose coordinates are the input - output pairs for *f* . In set notation. The graph is

$$\{(x, f(x)): x\in D\} .$$

EXAMPLE: Graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y).

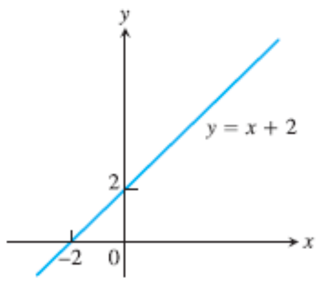
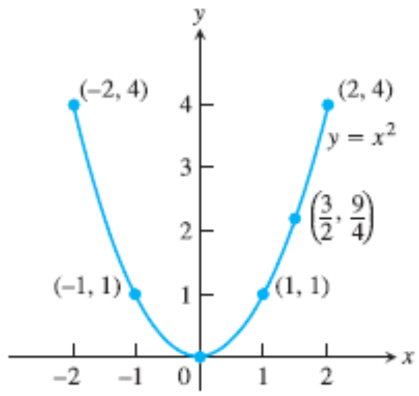


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

EXAMPLE 2: Graph the function $y=x^2$ over the interval [-2, 2].

Solution: Make a table of xy-pairs that satisfy the equation $y=x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.4).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



3.4 Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

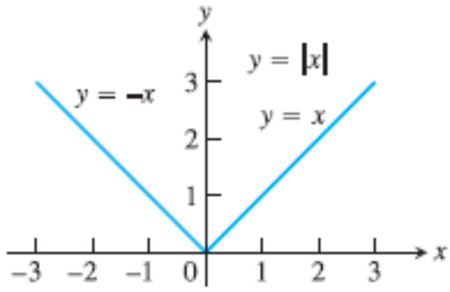


FIGURE 1.7 The absolute value function has domain (- ∞, ∞) and range [0, ∞).

EXAMPLE 3: The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Is is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$_when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.8).

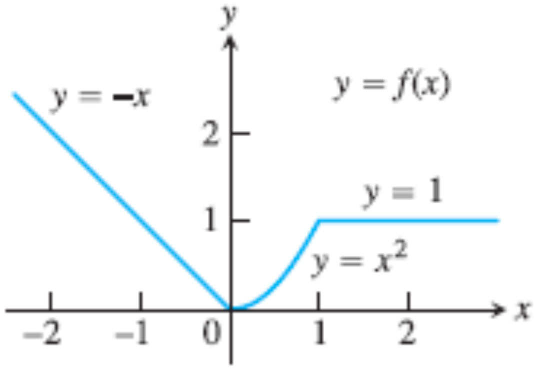
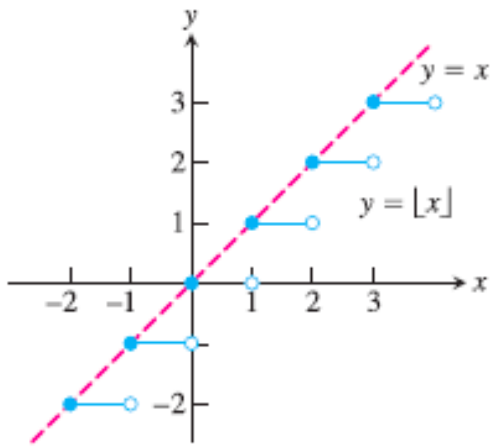


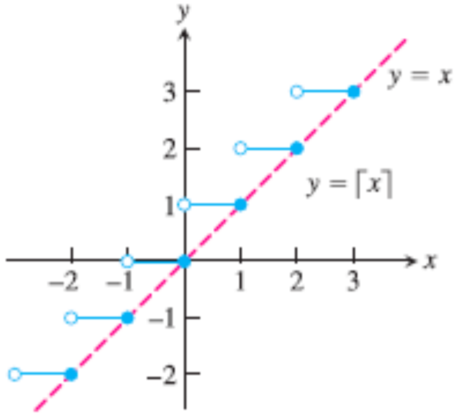
FIGURE 1.8 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 3).

EXAMPLE 4: The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.9 shows the graph. Observe that

$\lfloor 2,4 \rfloor = 2$, $\lfloor 1,9 \rfloor = 1$, $\lfloor 0 \rfloor = 0$, $\lfloor -1.2 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.2 \rfloor = 0$,$\lfloor -0.3 \rfloor = -1$, $\lfloor -2 \rfloor = -2$



EXAMPLE 5: The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function** . It is denoted $\lceil x \rceil$. Figure 1.10 shows the graph. For positive values of x, this function might represent.



3.5 Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The graph of an even function is **symmetric about they-axis**. Since *f(-x) = f(x)* .

The graph of an odd function is **symmetric about the origin**. Since *f(-x) = - f(x)* .

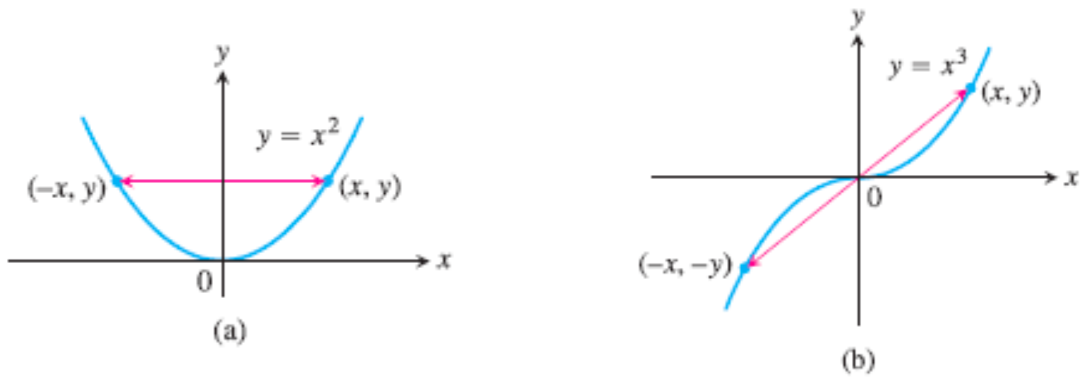


FIGURE 1.11 (a) The graph of *y=x*2 (an even function) is symmetric about the y-axis. (b) The graph *of y=x*^{3} (an odd function) is symmetric about the origin.

EXAMPLE 8

$f(x)=x^2$ Even function: $(-x)^2=x^2$ for all x; symmetry about y-axis.

$f(x)=x^2+1$ Even function: $(-x)^2+1=x^2+1$ for all x; symmetry about y-axis. (Figure1.12a).

$f(x) = x$ Odd function: $(-x) = -x$ for all x; symmetry about the origin.

$f(x) = x +1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.12b).

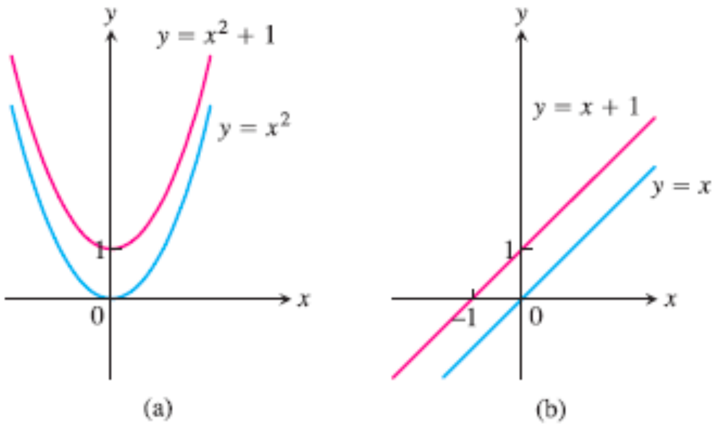


FIGURE 1.12 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2+1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost.

3.6 Common Functions:

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function.** Figure 1.13a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function.** Constant functions result when the slope $m = 0$ (Figure 1.13 b). A linear function with a positive slope whose graph passes through the origin is called a *proportionality* relationship.

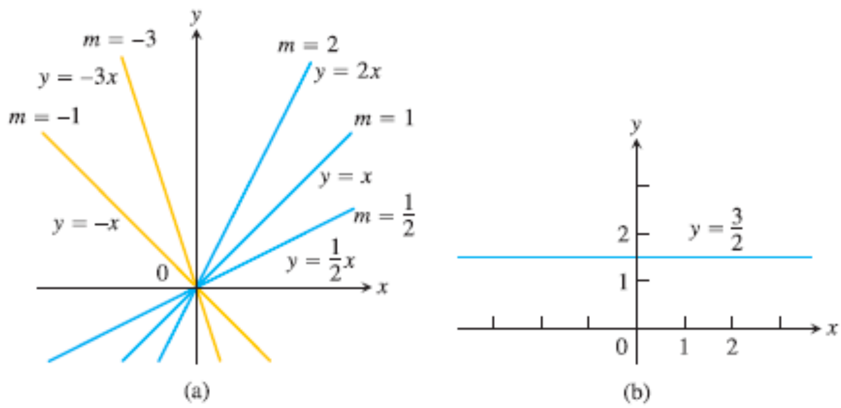


FIGURE 1.13 (a) Lines through the origin with slope *m*. (b) A constant function with slope *m* = 0.

Power Functions A function *f(x)* = *xa*, where *a* is a constant, is called a **power function.** There are several important cases to consider.

(a) *a* = *n,* a positive integer.

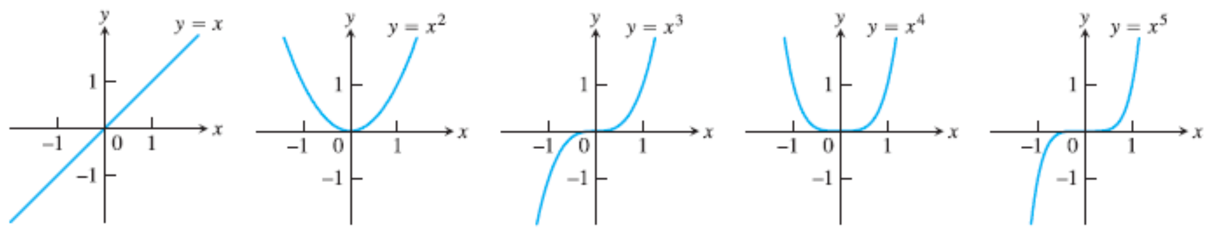
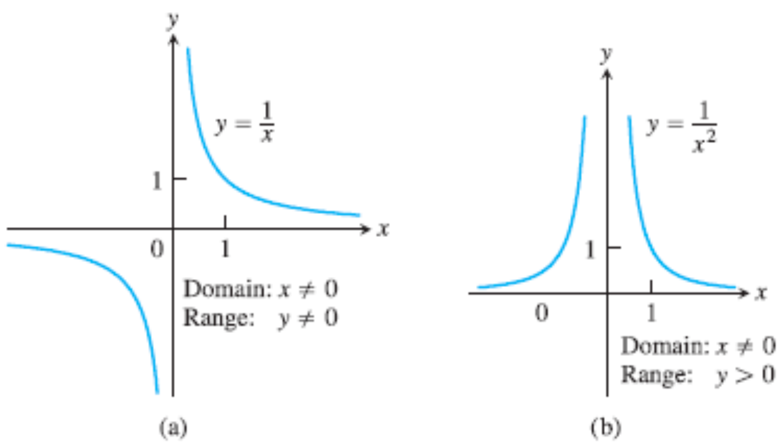


FIGURE 1.14 Graphs of *f(x)* = *xn* _, *n* = 1,2,3,4,5, defined for -∞ < *x* < ∞ .

(b) *a* = -*1 or a* = -2 .



(c) $a = 1/2$, $1/3$, $3/2$ *and* $2/3$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt{x}3$ are the **square root and cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.16, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$ **.**

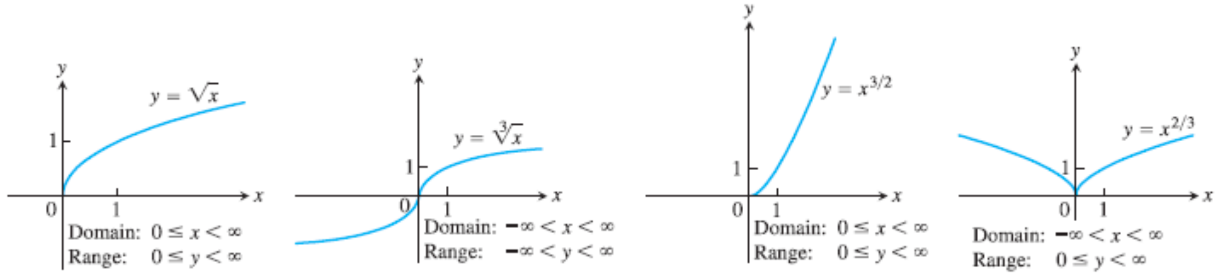


FIGURE 1.16 Graphs of the power functions $a = 1/2$, $1/3$, $3/2$ *and* $2/3$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where n is a nonnegative integer and the numbers a_0, a_1, a_2,…, a_n are real constants (called the **coefficients** of the polynomial) . All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions.** Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.17 shows the graphs of three polynomials.

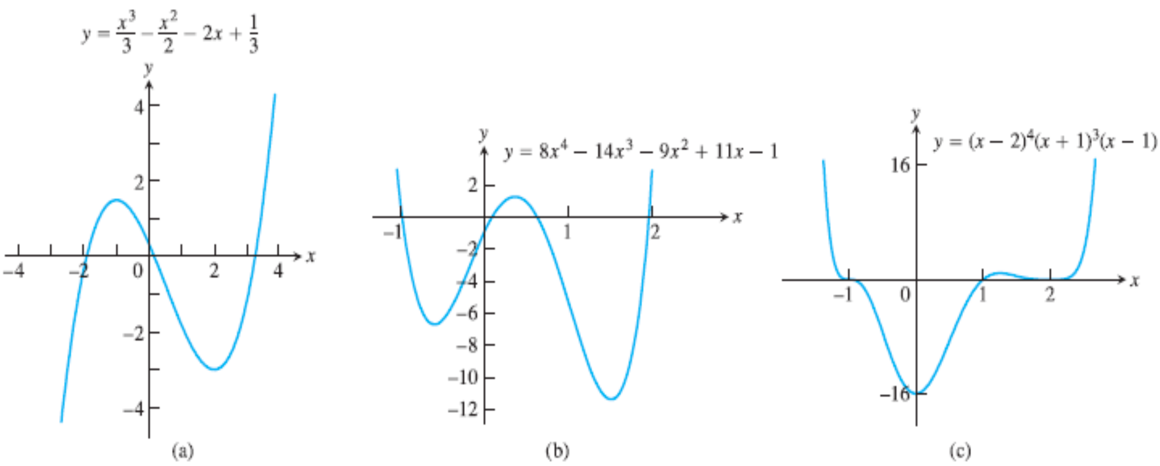


FIGURE 1.17 Graphs of three polynomial functions.

Rational Functions **A rational function** is a quotient or ratio *f(x) = p(x)/q(x),* where *p* and *q* are polynomials. The domain of a rational function is the set of all real *x* for which *q(x)* ≠ 0. The graphs of several rational functions are shown in Figure 1.18.

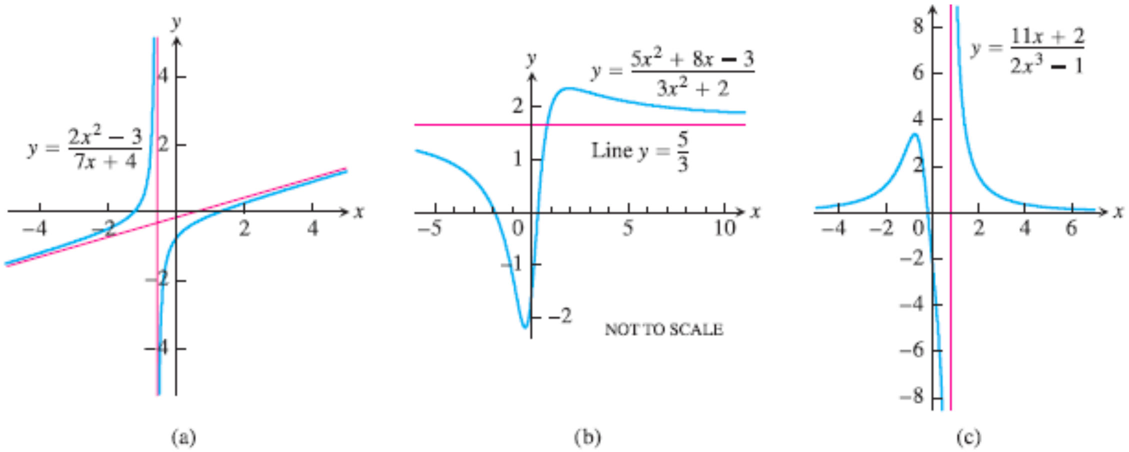


FIGURE 1.18 Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.

Exponential Functions Functions of the form *f(x) = ax,* where the base *a* > 0 is a positive constant and *a* of 1, are called **exponential functions.** All exponential functions have domain (- ∞, ∞) and range (0, ∞), so an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1.20.

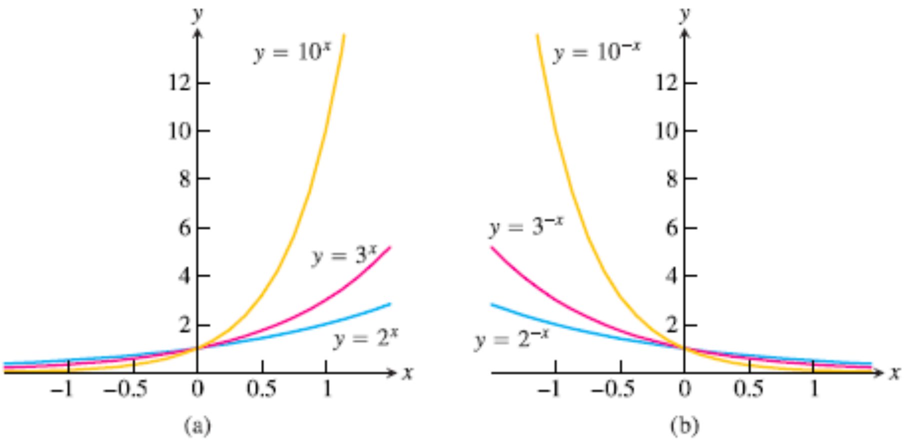


FIGURE 1.20 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant . Figure 1.21 shows the graphs of four logarithmic functions with various bases. In each case, the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

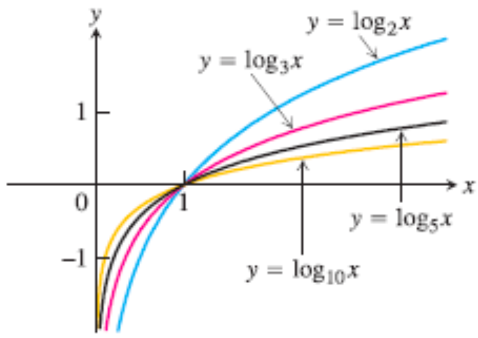


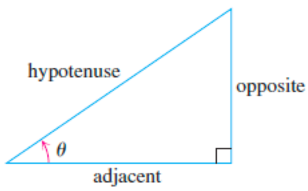
FIGURE 1.21 Graphs of four logarithmic functions

Trigonometric Functions

This section reviews the basic trigonometric functions. The trigonometric functions are important because they are periodic or repeating and therefore model many naturally occurring periodic processes.

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.22).

sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

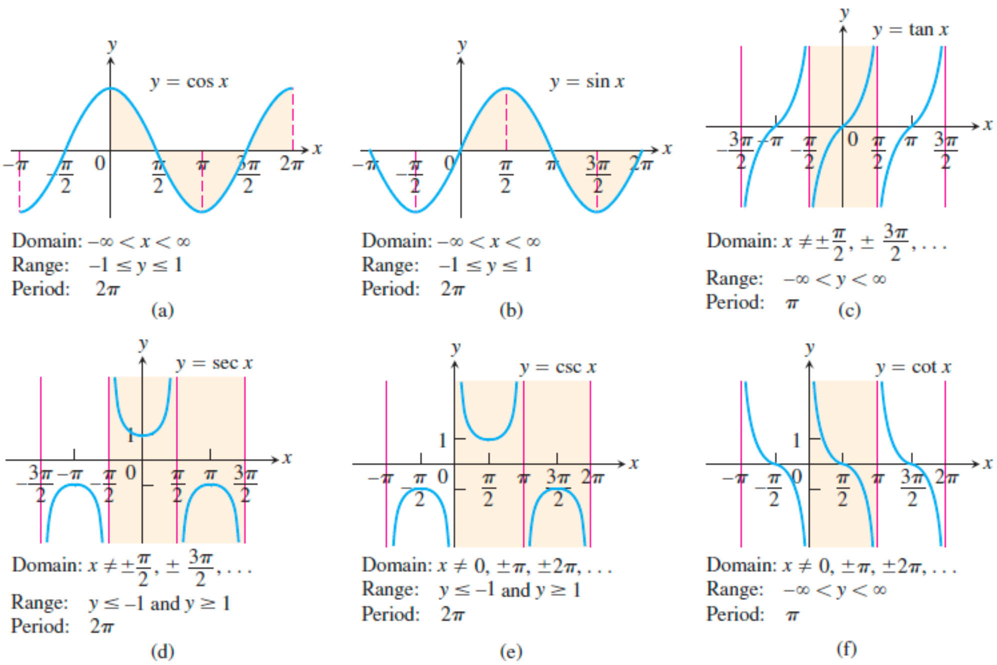
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Figure 17.



The symmetries in the graphs in Figure 1.23 reveal that the cosine and secant functions are even and the other four fuctions are odd:

Even	**Odd**
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Fundamental identities

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{3}$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{4}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{5}$$

HomeWorks :

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

2. Functions can be categorized as: power functions, polynomials (with degree stated), rational functions, algebraic functions, trigonometric functions, exponential functions, or logarithmic functions. Note that some functions may fit into multiple categories.

1. a. $f(x) = 7 - 3x$ b. $g(x) = \sqrt[5]{x}$
   c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ d. $r(x) = 8^x$
2. a. $F(t) = t^4 - t$ b. $G(t) = 5^t$
   c. $H(z) = \sqrt{z^3 + 1}$ d. $R(z) = \sqrt[3]{z^7}$
3. a. $y = \dfrac{3 + 2x}{x - 1}$ b. $y = x^{5/2} - 2x + 1$
   c. $y = \tan \pi x$ d. $y = \log_7 x$
4. a. $y = \log_5 \left(\dfrac{1}{t}\right)$ b. $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$
   c. $g(x) = 2^{1/x}$ d. $w = 5 \cos \left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$