3-Solutions of ordinary differential equations

3.1 Introduction and Simple Examples

Differential equations are a group of equations that contain derivatives, e.g.

$$\frac{dy}{dx} + 2xy^2 = 0. \tag{3.1}$$

This equation is sometimes written as $y' + 2xy^2 = 0$. The *general solution* of a differential equation is given by the set of all functions that *satisfy* the equation. E.g., all functions

$$y(x) = \frac{1}{x^2 + c},$$

with $c \in \mathbb{R}$ an arbitrary constant are solutions to the differential equation (3.1),

$$\text{l.h.s} = \frac{dy}{dx} + 2xy^2 = -\frac{1}{(x^2+c)^2} \cdot 2x + 2x \cdot \frac{1}{(x^2+c)^2} = 0 = \text{r.h.s.}$$

Note that "l.h.s." and "r.h.s." stand for left- and right-hand side, respectively. Differential equations occur everywhere in physics. Examples include:

- Simple harmonic oscillator: displacement x of a particle gives rise to a restoring force $F = -kx$, where $k > 0$ is the spring constant.

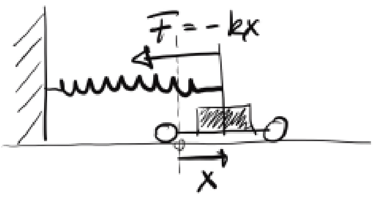


Figure 3.1: An example of a harmonic oscillator

According to Newton's law of motion: Force = mass × acceleration = $m\,dv/dt = m\,d^2x/dt^2$, so that

$$m\frac{d^2x}{dt^2} = -kx. \tag{3.2}$$

- Newton's law of cooling:

Figure 3.2: A hot cup of tea with temperature $T > T_S$.

This law states that the rate at which a hot body cools is proportional to the difference $T - T_S$ between the temperature T of the body and the temperature T_S of the surroundings. Expressed as an equation, this says:

$$dT/dt = -\alpha\,(T - T_S)\,, \tag{3.3}$$

where α is a positive constant.

- Quantum mechanics: in order to determine the allowed energies E of a quantum system, we have to solve the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{3.4}$$

- Wave equation: in vacuum, the change of the electric field $E(x,t)$ as a function of position x and time t is described by the wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2\frac{\partial^2 E}{\partial x^2}. \tag{3.5}$$

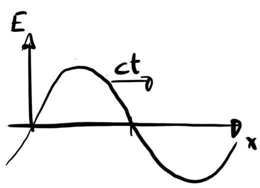


Figure 3.3: An electromagnetic wave propagating with velocity c. It is easy to check that $E(x,t) = f(x - ct)$ with f an arbitrary function is a solution of the wave equation.

These are all *differential equations*, because they contain derivatives d^2x/dt^2, dT/dt, $d^2\psi/dx^2$, $\partial^2 E/\partial t^2$, and $\partial^2 E/\partial x^2$. We write down a differential equation (usually based upon some assumptions about a physics system) and then try to find the functions that satisfy the equation. In physics, once we have such a function, we can use it to predict other behaviours of the system. The aim of this and the next few lectures is to explain how to find solutions of differential equations.

3.1.1 Terminology

We need to define some terminology that will often be used:

- Independent and dependent variables: For the harmonic oscillator, the displacement $x(t)$ *depends* on time t, so we call t the independent variable and x the dependent variable. The idea is to find the function $x(t)$ expressing how the dependent variable depends on the independent variable. Similarly, for the cooling body, time t is the independent variable and T is the dependent variable. For the wave equation the electric field $E(x, t)$ is the dependent variable, x and t are independent variables.
- Ordinary differential equations (ODE's): these are equations with *only one independent variable*, so that we only have ordinary differentials (e.g. d^2x/dt^2), not partial differentials. Examples (3.1), (3.2), (3.3), and (3.4) are all ODE's. The wave equation (3.5) is a *partial differential equation.* In this course we will only discuss ODE's.
- Order of differential equation: this refers to the maximum number of times that the dependent variable is differentiated in the equation. In examples (3.1) and (3.3) we only have first derivatives, so these are *first-order* differential equations. The harmonic oscillator (3.2) is an example of a *second-order* differential equation since $x(t)$ is differentiated twice (d^2x/dt^2).
- Linearity: a differential equation is *linear* if the *dependent* variable occurs at most to the first power. Examples include (3.2), (3.3), and (3.4). Example (3.1) is a *non-linear* differential equation since the dependent variable y is squared. Some other examples:

  (a) $\frac{dy}{dx} = \cot y$ (not linear because of term $\cot y$)

  (b) $y\frac{dy}{dx} = 1$ (not linear because of product yy')

  (c) $x^2 y + \sin x \frac{d^2y}{dx^2} = x^5$ (linear because dependent variable y only to first power)

  The general form of an n-th order *linear* ODE is given by

  $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \quad (3.6)$$

  where $a_i(x)$ and $b(x)$ are functions of x (could also be constant) and $a_n(x) \neq 0$.

- Homogeneity: a linear ODE is *homogeneous* if the dependent variable appears to the *first power in every term.* For example, the harmonic oscillator ODE (3.2) is homogeneous, because every term contains $x(t)$.

  The general n-th order linear ODE (3.6) is *homogeneous* $\Leftrightarrow b(x) \equiv 0$.

- Notation: in physics, the dependent and independent variables are often given symbols which reflect the physical meaning of the variables (e.g. T for temperature, t for time). But in these lectures, we will usually call the dependent variable y and the independent variable x (as in (3.1)).

3.1.2 A Simple Example

Defining $\theta := T - T_S$ as a new dependent variable, we can rewrite Newton's law of cooling (3.3) as

$$d\theta/dt = -\alpha\theta. \tag{3.7}$$

The reason is that

$$\frac{d\theta}{dt} = \frac{d}{dt}(T - T_S) = \frac{dT}{dt}.$$

The ODE (3.7) is first order, linear, and homogeneous. One solution of this equation is

$$\theta(t) = e^{-\alpha t},$$

since $d\theta/dt = e^{-\alpha t} \cdot (-\alpha) = -\alpha\theta$. However, it is not the only possible solution. A more general solution is

$$\theta(t) = Ae^{-\alpha t}. \tag{3.8}$$

The arbitrary constant $A \in \mathbb{R}$ multiplies the whole solution because the original ODE is *homogeneous.*

Two important messages:

- The general solution of an ODE contains arbitrary constants (integration constants). We will see that 1st-order ODE's always lead to one arbitrary constant, and 2nd-order ODE's lead to two arbitrary constants.
- For a linear homogeneous ODE, if we have found a solution $y_0(x)$ then the function $y(x) = Ay_0(x)$, where $A \in \mathbb{R}$ is an arbitrary constant, is also a solution.

Proof of 2nd statement: We consider a general homogeneous linear ODE, Eq. (3.6) with $b(x) \equiv 0$. Let $y_0(x)$ be a solution of this ODE,

$$a_n(x)\frac{d^n y_0}{dx^n} + \ldots + a_1(x)\frac{dy_0}{dx} + a_0(x)y_0 = 0. \quad (*)$$

We now show that $y(x) = Ay_0(x)$ also satisfies the ODE,

$$\begin{aligned}
& a_n(x)\frac{d^n y}{dx^n} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y \\
\overset{y(x)=Ay_0(x)}{=} \quad & a_n(x)A\frac{d^n y_0}{dx^n} + \ldots + a_1(x)A\frac{dy_0}{dx} + a_0(x)Ay_0 \\
= \quad & A\left(a_n(x)\frac{d^n y_0}{dx^n} + \ldots + a_1(x)\frac{dy_0}{dx} + a_0(x)y_0\right) \\
\overset{(*)}{=} \quad & A \cdot 0 = 0.
\end{aligned}$$

3.1.3 Fixing the Arbitrary Constants

The ODE itself does not contain the information needed to fix the values of any arbitrary constants that appear. But in real physics situations, there is always additional information that fixes them. This additional information is referred to as *initial conditions* or *boundary conditions*. We illustrate this for our previous examples:

- Newton's law of cooling. The physical quantity $\theta(T) = T(t) - T_S$, representing the temperature difference between the hot body and the surroundings is given by

  $$\theta(t) = Ae^{-\alpha t},$$

  where $A \in \mathbb{R}$ denotes the arbitrary integration constant. In this example, we might know the initial value of θ, i.e. the temperature $T(0)$ of the hot body at $t = 0$, from which we could calculate $T(0) - T_S = \theta(0)$. From the general solution of the ODE we know that $\theta(0) = A$, so this fixes the value of A. This is an example of an initial condition.

- Harmonic oscillator. In this case, the general solution is given by

  $$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{3.9}$$

  where $\omega = \sqrt{k/m}$, and A and B are arbitrary constants. Note that the general solution contains two integration constants because the ODE (3.2) is 2nd-order. Let us quickly verify that (3.9) indeed satisfies the ODE (3.2),

  $$\frac{dx}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t) = v(t), \tag{3.10}$$

  $$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) = -\omega^2 x. \tag{3.11}$$

  Eq. (3.11) is indeed equivalent to (3.2), as required. To fix two arbitrary constants we need two initial conditions, e.g. the position x_0 and the velocity v_0 at $t = 0$. Plugging this into the general solution $x(t)$ (3.9) and the corresponding velocity $v(t) = dx/dt$ (3.10), we obtain $x(0) = A = x_0$ and

  $$v(0) = B\omega = v_0 \Longleftrightarrow B = \frac{v_0}{\omega}.$$

  Hence the *special solution* that satisfies the initial conditions is given by

  $$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t).$$

3.2 Separable First-Order ODE's

There is a large class of first-oder ODE's that are simple to solve, because they have a special property called "separability". To explain separability, take the following example: we want to solve the equation

$$\frac{dy}{dx} = \left(1 + x^2\right) y. \tag{3.12}$$

Formally, we can rearrange this to give

$$\frac{dy}{y} = \left(1 + x^2\right) dx,$$

separating the dependent and independent variables. Now integrate both sides:

$$\int \frac{1}{y} dy = \int \left(1 + x^2\right) dx.$$

Noting that these are *indefinite* integrals, we obtain

$$\ln |y| = x + \frac{1}{3}x^3 + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Exponentiating this equation we obtain

$$|y| = \exp\left(x + \frac{1}{3}x^3 + C\right).$$

From this it follows that

$$y = \pm e^C \exp\left(x + \frac{1}{3}x^3\right) = A \exp\left(x + \frac{1}{3}x^3\right), \tag{3.13}$$

where in the last step we have redefined the integration constant as $A := \pm e^C$. By definition, this constant can take any positive or negative value. Since $y \equiv 0$ is a trivial solution of the ODE (3.12), we can generalise this to $A \in \mathbb{R}$. Such a multiplicative integration constant is expected since the original ODE is homogeneous. We check if (3.13) is indeed the general solution of (3.12),

$$\frac{dy}{dx} = A \exp\left(x + \frac{1}{3}x^3\right)\left(1 + x^2\right) = \left(1 + x^2\right) y,$$

which agrees with the original ODE.

> General rule: Take the original ODE, splitting dy/dx into dy and dx. If we can rearrange the equation so that dy and all other quantities containing y are on the left and dx and all quantities containing x are on the right, then the ODE is called *separable*. In this case the general solution can be found by integrating the separated equation. Note that the integrals are indefinite. This is where the arbitrary integration constant enters.

Warning: One cannot always just pick apart a differential like this. In general it might help to remember that it is a limit, and that the dy and dx belong together. If the operation makes sense when you move away from the limit and then move back, its usually ok. If we were mathematicians, we'd have to prove this formally, of course.

3.2.1 Worked Examples

(1) Find the general solution $y = f(x)$ of the differential equation

$$\frac{dy}{dx} = y \sin x.$$

This ODE is separable,

$$\frac{dy}{y} = \sin x \, dx$$

We integrate both sides,

$$\int \frac{1}{y} dy = \int \sin x \, dx$$

and perform the indefinite integrals,

$$\ln|y| = -\cos x + C \qquad (C \in \mathbb{R})$$

Exponentiating this equation,

$$|y| = e^{-\cos x + C} = e^{C} e^{-\cos x}$$

$$\Rightarrow y = \pm e^{C} e^{-\cos x}$$

$$= A e^{-\cos x} \quad \text{(redefine integration constant)}$$

Here $A \in \mathbb{R}$ can be an arbitrary constant.
We check if our solution indeed satisfies the ODE,

$$\frac{dy}{dx} = A e^{-\cos x} \cdot \sin x = \sin x \, y$$

as required.

(2) Find the solution of the separable ODE

$$x\frac{dy}{dx} - xy = y,$$

for which $y = 1$ when $x = 2$.

We first separate variables,

$$x\frac{dy}{dx} = y(1+x) \iff \frac{dy}{y} = \left(1+\frac{1}{x}\right)dx$$

Integrating both sides:

$$\int \frac{1}{y}\,dy = \int \left(1+\frac{1}{x}\right)dx$$

$$\Rightarrow \ln|y| = x + \ln|x| + C$$

Exponentiating both sides:

$$|y| = e^{C}|x|e^{x}$$

$$\Rightarrow y = \underbrace{\pm e^{C}}_{=:A} x e^{x}$$

We check if our solution satisfies the ODE,

$$x\frac{dy}{dx} - xy = x\left(Ae^{x} + Axe^{x}\right) - xAxe^{x}$$

$$= Axe^{x} = y \quad \checkmark$$

We use the initial condition to determine A:

$$1 = 2Ae^{2} \iff A = \frac{1}{2e^{2}}$$

(3) The two above examples were homogeneous, linear ODE's, giving rise to a multiplicative integration constant. Let's now solve the non-linear first-order ODE

$$\frac{dy}{dx} - xy^2 = x.$$

We separate variables,

$$\frac{dy}{dx} = x(1+y^2) \iff \frac{dy}{1+y^2} = x\,dx$$

Integration yields

$$\int \frac{1}{1+y^2}\,dy = \int x\,dx$$

$$\Rightarrow \quad \arctan y = \frac{1}{2}x^2 + c \quad ; \; c \in \mathbb{R}$$

$$\Rightarrow \quad y = \tan\left(\frac{1}{2}x^2 + c\right)$$

We check if our solution satisfies the ODE:

$$\begin{aligned} \frac{dy}{dx} - xy^2 &= \left[1 + \tan^2\left(\tfrac{1}{2}x^2 + c\right)\right] \cdot x \\ &\quad - x\tan^2\left(\tfrac{1}{2}x^2 + c\right) \\ &= x \end{aligned}$$

as required.

3.3 Linear First-Order ODE's

Previously, I explained how to solve first-order ODE's in the case where they are separable. Unfortunately, most first-order ODE's are not separable. However, many of them can still be solved. The aim of this section is to explain a completely general method for solving *linear* first-order ODE's. According to Eq. (3.6), the most general form of such an ODE is

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

Dividing by $a_1(x)$ we can bring this to the "standard form" of a linear first-order ODE:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{3.14}$$

$P(x)$ and $Q(x)$ can be complicated functions of x. Note that this linear ODE is inhomogeneous unless $Q(x) \equiv 0$.

3.3.1 'Integrating Factor' Method

The idea is to multiply the ODE (3.14) with an appropriate *integrating factor* $S(x)$,

$$S(x)\frac{dy}{dx} + S(x)P(x)y = S(x)Q(x). \tag{3.15}$$

If we demand that $S(x)$ satisfies the equation

$$\frac{dS}{dx} = S(x)P(x), \tag{3.16}$$

we can rewrite the l.h.s. of (3.15) as

$$S(x)\frac{dy}{dx} + S(x)P(x)y = S(x)\frac{dy}{dx} + \frac{dS}{dx}y = \frac{d}{dx}\left[S(x)y\right],$$

where in the last step we have used the product rule of differentiation. Our ODE (3.15) then reads

$$\frac{d}{dx}\left[S(x)y\right] = S(x)Q(x),$$

and can easily be integrated,

$$S(x)y = \int S(x)Q(x)dx + C, \tag{3.17}$$

with $C \in \mathbb{R}$. The integrating factor is determined by Eq. (3.16), which is a *separable* first-order ODE,

$$\frac{dS}{S} = P(x)dx,$$

After integration this gives

$$\ln|S| = \int P(x)dx + B.$$

with $B \in \mathbb{R}$. Exponentiating both sides we obtain

$$S(x) = e^{\int P(x)dx}. \tag{3.18}$$

Note that we have fixed the integration constant since we just need a special solution of (3.16). Combining Eqs. (3.17) and (3.18), we have found the general solution of the linear first-order ODE (3.14),

$$y = \frac{1}{S(x)}\left(\int S(x)Q(x)dx + C\right), \quad \text{where} \quad S(x) = e^{\int P(x)dx}. \tag{3.19}$$

Rather than memorising the formal solution it is better just to remember the basic idea. Let's look at the example

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{3.20}$$

We multiply the ODE with the integrating factor $S(x)$,

$$S(x)\frac{dy}{dx} + S(x)\frac{1}{x}y = S(x) \quad (*)$$

and demand that it satisfies the condition

$$\frac{dS}{dx} = S(x)\frac{1}{x}.$$

In this case we can write the l.h.s. of (*) as $\frac{d}{dx}[S(x)y]$. The ODE for $S(x)$ can be easily solved by separation of variables,

$$\frac{dS}{S} = \frac{dx}{x} \Longrightarrow \ln|S| = \ln|x| + B \Longrightarrow S(x) = Ax \quad (A \in \mathbb{R}).$$

Note that we just need a special solution so we can set $A = 1$. With $S(x) = x$ we can write (*) as

$$\frac{d}{dx}(xy) = x,$$

which after integration gives

$$xy = \frac{1}{2}x^2 + C \Longrightarrow y = \frac{1}{2}x + \frac{C}{x}.$$

This is the general solution of ODE (3.20).

3.3.2 Worked Examples

(1) Find the general solution of the linear first-order ODE

$$x^2 \frac{dy}{dx} - 2xy = \frac{1}{x}.$$

We bring the ODE to standard form:

$$\frac{dy}{dx} - \frac{2}{x} y = \frac{1}{x^3}$$

and multiply with an integrating factor $S(x)$,

$$S(x)\frac{dy}{dx} - S(x)\frac{2}{x} y = S(x)\frac{1}{x^3}$$

and demand $\frac{dS}{dx} = -S(x) \cdot \frac{2}{x}$ (1)

Our ODE then reads

$$\frac{d}{dx}\left[S(x) y \right] = S(x) \frac{1}{x^3}. \quad (2)$$

Obtain $S(x)$ by separation of variables of (1):

$$\frac{dS}{S} = -\frac{2}{x} dx \Rightarrow \ln|S| = -2\ln|x|$$

$$\Rightarrow S(x) = \frac{1}{x^2}$$

$$\overset{(2)}{\Rightarrow} \frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^5}$$

$$\Rightarrow \frac{y}{x^2} = -\frac{1}{4} x^{-4} + C \quad (C \in \mathbb{R})$$

$$\Rightarrow y(x) = -\frac{1}{4x^2} + Cx^2$$

(2) Find the general solution of the linear first-order ODE

$$\frac{dy}{dx} + y\cos x = \sin(2x).$$

Multiply ODE with integrating factor $S(x)$

$$S(x)\frac{dy}{dx} + \underbrace{S(x)\cos x}_{\stackrel{!}{=}\frac{dS}{dx}\ (1)}\, y = S(x)\sin(2x)$$

$$\Rightarrow \frac{d}{dx}\left(S(x)y\right) = S(x)\sin(2x) \quad (2)$$

$$\frac{dS}{S} = \cos x\, dx \Rightarrow \ln|S| = \sin x$$

$$\Rightarrow S(x) = e^{\sin x}$$

$$\rightarrow (2) \quad \frac{d}{dx}\left[e^{\sin x} y\right] = e^{\sin x}\sin(2x)$$

$$\Rightarrow e^{\sin x} y = \int dx\, e^{\sin x}\sin(2x) + C$$

$$= 2\int dx\, e^{\sin x}\cos x \sin x + C$$

$$\underset{\substack{\xi = \sin x \\ d\xi = \cos x\, dx}}{=} 2\int d\xi\, \underset{u'}{e^{\xi}}\, \underset{v}{\xi} + C$$

$$= 2\left[e^{\xi}\xi - \int d\xi\, e^{\xi}\right] + C$$

$$= 2(\xi - 1)e^{\xi} + C$$

$$= 2(\sin x - 1)e^{\sin \xi} + C$$

$$\Rightarrow y(x) = 2(\sin x - 1) + C e^{-\sin x}$$

3.4 Perfect Differential Method

If a first-order ODE is non-linear (so it contains terms like y^2, $yy', \ldots$), the systematic method discussed in Sec. 3.3 cannot be used. If the ODE is also non-separable, then the only realistic hope left is the *perfect-differential method*, also known as the *exact-differential method*. If the ODE contains dy/dx only to the first power, then it can always be written as

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0, \tag{3.21}$$

or equivalently as

$$P(x,y)dx + Q(x,y)dy = 0. \tag{3.22}$$

Here P and Q are functions of x and y. Now suppose that $P(x,y)$ and $Q(x,y)$ are the partial differentials with respect to x and y of some other function $f(x,y)$,

$$P(x,y) = \frac{\partial f}{\partial x}\,, \qquad Q(x,y) = \frac{\partial f}{\partial y}\,. \tag{3.23}$$

In this case, the l.h.s. of Eq. (3.22) can be written as the total differential of f,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}dy = P(x,y)dx + Q(x,y)dy = 0. \tag{3.24}$$

Since $df = 0$, the function f has to be constant,

$$f(x,y) = C, \tag{3.25}$$

where $C \in \mathbb{R}$ is an arbitrary constant. This represents the general solution to the original ODE (3.22). Note that Eq. (3.25) does not contain derivatives. It implicitly defines the functions $y(x)$ that satisfy the original ODE. However, it is not always possible to rearrange the implicit expression (3.25) to express y as a function of x (to solve for y).

In this method, we need to test whether the given $P(x,y)$ and $Q(x,y)$ can be represented as $\partial f/\partial x$ and $\partial f/\partial y$. A *necessary* condition for this to be true is

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}. \tag{3.26}$$

It can be shown that $\partial P/\partial y = \partial Q/\partial x$ is also a *sufficient* condition for P and Q to be representable in this way.

To see how this works in practice, say we want to find the general solution of the non-linear, non-seperable, first-order ODE

$$2x^2y\frac{dy}{dx} + 2xy^2 = 1. \tag{3.27}$$

This ODE can be expressed in the standard form (3.22) with

$$P(x,y) = 2xy^2 - 1, \qquad Q(x,y) = 2x^2y.$$

We first need to find out if the ODE can be solved by the perfect differential method. We apply the standard test: since $\partial P/\partial y = 4xy = \partial Q/\partial x$, we know that there exists a function $f(x,y)$ such that $P(x,y) = \partial f/\partial x$ and $Q(x,y) = \partial f/\partial y$,

$$\text{I.}\ \frac{\partial f}{\partial x} = 2xy^2 - 1, \quad \text{II.}\ \frac{\partial f}{\partial y} = 2x^2 y.$$

Integrating the first equation, we obtain

$$f(x,y) = \int (2xy^2 - 1)dx = x^2y^2 - x + g(y), \tag{3.28}$$

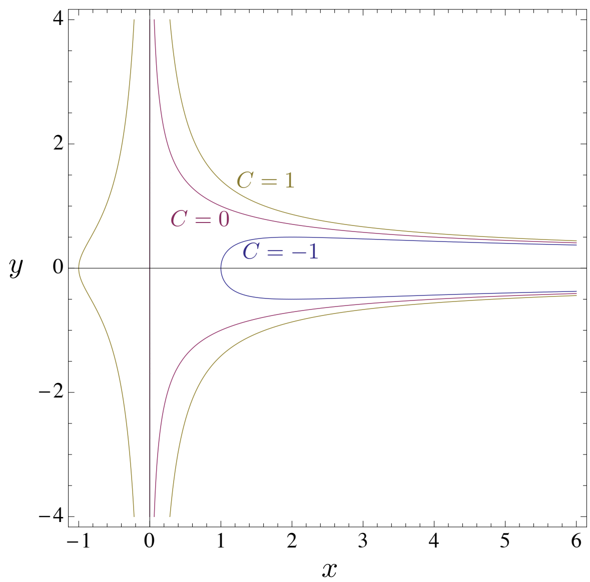
where $g(y)$ is the "integration constant". Note that y is held constant in the above integration, so g can be function of y. Or to phrase this differently, taking the partial derivative with respect to x in Eq. I, any function of y is treated as a constant and drops out. Likewise, from integration of II, we obtain

$$f(x,y) = \int 2x^2 y\, dy = x^2y^2 + h(x). \tag{3.29}$$

We have to determine the functions $g(y)$ and $h(x)$ such that Eqs. (3.28) and (3.29) give us the same expression for $f(x,y)$. This is the case for $h(x) = -x$ and $g(y) = 0$, corresponding to $f(x,y) = x^2y^2 - x$. The general solution of ODE (3.26) is given by $f(x,y) = \text{const}$,

$$x^2y^2 - x = C, \tag{3.30}$$

with $C \in \mathbb{R}$. Contours of points (x,y) that satisfy this equation for different values of the integration constant C are shown below. Note that we can solve Eq. (3.30) for y, $y = \pm\sqrt{1/x + C/x^2}$.



3.4.1 Worked Example

Show that the ODE

$$\left(2xy + e^{-x^2}\right)\frac{dy}{dx} = 2xye^{-x^2} - y^2$$

can be written as an exact differential and find the general solution in an implicit form.

We first bring this ODE into the standard form:

$$\underbrace{(y^2 - 2xye^{-x^2})}_{=:P(x,y)}dx + \underbrace{(2xy + e^{-x^2})}_{=:Q(x,y)}dy = 0. \quad (*)$$

Can this ODE be written as an exact differential?

$$\frac{\partial P}{\partial y} = 2y - 2xe^{-x^2} = \frac{\partial Q}{\partial x} \quad \text{yes!}$$

→ There exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = P(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = Q(x,y)$$

and the general solution of (*) is given by $f(x,y) = C$ with $C \in \mathbb{R}$. Let's determine $f(x,y)$:

I. $\frac{\partial f}{\partial x} = y^2 - 2xye^{-x^2} \Rightarrow f(x,y) = xy^2 + ye^{-x^2} + g(y)$

II. $\frac{\partial f}{\partial y} = 2xy + e^{-x^2} \Rightarrow f(x,y) = xy^2 + ye^{-x^2} + h(x)$

I and II are consistent if $h(x) = g(y) = 0$ and $f(x,y) = xy^2 + ye^{-x^2}$.

→ The general solution of (*) is given by

$$xy^2 + ye^{-x^2} = C, \quad C \in \mathbb{R}.$$

3.5 Second-Order Linear ODE's with Constant Coefficients

Second-order ODE's are those containing the second derivative d^2y/dx^2 of the dependent variable. The only kind of second-order ODE's considered here are *linear* ODE's, in which the coefficients of d^2y/dx^2, dy/dx and y are all constant,

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = b(x),$$

where $b(x)$ is a function of x only. We will assume that the constants a_2, a_1 and a_0 are real numbers and that $a_2 \neq 0$ (otherwise the ODE would be first order). Dividing by a_2, the ODE can always be brought to the *standard form*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x). \tag{3.31}$$

3.5.1 Homogeneous ODE's

We start by the discussing the special case of *homogeneous* ODE's, for which $f(x) = 0$,

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0. \tag{3.32}$$

Such ODE's are solved by functions of the form

$$y = e^{kx}, \tag{3.33}$$

provided that the constant k is chosen appropriately. To see this we first calculate the first and second derivatives of the function (3.33),

$$\frac{dy}{dx} = ke^{kx}, \quad \frac{d^2y}{dx^2} = k^2e^{kx}.$$

Inserting the ansatz (3.33) into the ODE (3.32) we obtain

$$0 = \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = k^2e^{kx} + pke^{kx} + qe^{kx} = \left(k^2 + pk + q\right) e^{kx}.$$

Our ansatz satisfies the ODE if the constant k is a solution of the quadratic equation

$$k^2 + pk + q = 0. \tag{3.34}$$

In general, there are two roots (solutions) of this equation, given by

$$k_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} \tag{3.35}$$

We have found two specific solutions $y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$. How to construct the general solution of the homogeneous ODE (3.32)? In general, if two functions $y_1(x)$ and $y_2(x)$ satisfy (3.32), then the *linear combination* $y(x) = Ay_1(x) + By_2(x)$ is also a solution (note that this is true for any linear homogeneous ODE):

$$\begin{aligned}\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy &= A\frac{d^2y_1}{dx^2} + B\frac{d^2y_2}{dx^2} + p\left(A\frac{dy_1}{dx} + B\frac{dy_2}{dx}\right) + q\left(Ay_1 + By_2\right) \\ &= A\underbrace{\left(\frac{d^2y_1}{dx^2} + p\frac{dy_1}{dx} + qy_1\right)}_{=0} + B\underbrace{\left(\frac{d^2y_2}{dx^2} + p\frac{dy_2}{dx} + qy_2\right)}_{=0} = 0.\end{aligned}$$

Therefore, the function

$$y = Ae^{k_1 x} + Be^{k_2 x}, \tag{3.36}$$

with k_1, k_2 given in Eq. (3.35) is a solution of the homogeneous equation (3.32). This is the *general solution* since it contains two integration constants A, B, as required for a second-order ODE.

Let us now consider three possible types of roots k_1 and k_2:

- Real roots:

  $$\frac{p^2}{4} - q > 0$$

  In this case $k_1, k_2 \in \mathbb{R}$, $k_1 \neq k_2$, and the general solution (3.36) is the sum of two exponential functions. The function y is real if $A, B \in \mathbb{R}$.

- Complex roots:

  $$\frac{p^2}{4} - q < 0$$

  In this case we can write

  $$k_{1,2} = \alpha \pm i\beta, \tag{3.37}$$

  with $\alpha = -p/2 \in \mathbb{R}$ and $\beta = \sqrt{q - p^2/4} \in \mathbb{R}$. The general solution of the homogeneous ODE is given by

  $$\begin{aligned} y &= Ae^{k_1 x} + Be^{k_2 x} \\ &= Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x} \\ &= e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right) \\ &= e^{\alpha x}\left[(A+B)\cos(\beta x) + i(A-B)\sin(\beta x)\right], \end{aligned}$$

  where, in the last step, we have used Euler's Theorem $e^{iz} = \cos z + i \sin z$. We can always redefine the integration constants, $C := A + B$ and $D := i(A - B)$, yielding

  $$y = e^{\alpha x}\left[C\cos(\beta x) + D\sin(\beta x)\right]. \quad (C, D \in \mathbb{R}) \tag{3.38}$$

- Degenerate roots:

$$\frac{p^2}{4} - q = 0$$

The two roots of Eq. (3.35) are *identical*, $k_1 = k_2 = k = -p/2$. While $y_1 = e^{kx}$ is a solution of the ODE (3.32), $y = Ae^{kx}$ cannot be the most general solution since it only contains one arbitrary constant. We show that $y_2 = xe^{kx}$ is an independent second solution,

$$\begin{aligned}
\frac{d^2y_2}{dx^2} + p\frac{dy_2}{dx} + qy_2 &= \frac{d}{dx}\left(e^{kx} + kxe^{kx}\right) + p\left(e^{kx} + kxe^{kx}\right) + qxe^{kx} \\
&= ke^{kx} + ke^{kx} + k^2xe^{kx} + pe^{kx} + pkxe^{kx} + qxe^{kx} \\
&= 2\left(k + \frac{p}{2}\right)e^{kx} + \left(k^2 + pk + q\right)xe^{kx} \\
&\stackrel{k=-\frac{p}{2}}{=} \underbrace{\left(-\frac{p^2}{4} + q\right)}_{=0} xe^{kx} = 0.
\end{aligned}$$

Hence the general solution in the degenerate case (sometimes referred to as "marginal case") is

$$y = Ae^{kx} + Bxe^{kx}. \quad (A, B \in \mathbb{R}) \tag{3.39}$$

3.5.2 Worked Examples

Find the general solutions of the following homogeneous second-order differential equations:

$$\text{(a)}\ 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0, \quad \text{(b)}\ \frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0, \quad \text{(c)}\ \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

(a) We first bring this ODE into the standard form,

$$\frac{d^2y}{dx^2} + \frac{5}{2}\frac{dy}{dx} + \frac{3}{2}y = 0,$$

and insert the ansatz $y = e^{kx}$:

$$k^2 e^{kx} + \frac{5}{2} k e^{kx} + \frac{3}{2} e^{kx} = 0$$

$$\Rightarrow \quad k^2 + \frac{5}{2}k + \frac{3}{2} = 0$$

$$\Rightarrow \quad k_{1/2} = -\frac{5}{4} \pm \sqrt{\frac{25}{16} - \frac{3}{2}} = -\frac{5}{4} \pm \frac{1}{4}$$

We therefore find to real roots $k_1 = -1$ and $k_2 = -\frac{3}{2}$. The general solution of the ODE is

$$y = Ae^{-x} + Be^{-3/2x} \quad \text{with } A, B \in \mathbb{R}.$$

(b) Inserting the ansatz $y = e^{kx}$ we obtain

$$k^2 e^{kx} - 10 k e^{kx} + 25 e^{kx} = 0$$

$$\Rightarrow \quad k^2 - 10k + 25 = 0$$

$$\Rightarrow \quad k_{1/2} = 5 \pm \sqrt{25-25} = 5$$

We have two degenerate roots. In this case two independent solutions are given by $y_1 = e^{5x}$ and $y_2 = x e^{5x}$ and the general solution of the ODE is

$$y = A e^{5x} + B x e^{5x}$$

$$= (A + Bx) e^{5x} \quad \text{with } A, B \in \mathbb{R}.$$

(c) We insert the ansatz $y = e^{kx}$,

$$k^2 e^{kx} + 4 k e^{kx} + 5 e^{kx} = 0$$

$$\Rightarrow \quad k^2 + 4k + 5 = 0$$

$$\Rightarrow \quad k_{1/2} = -2 \pm \sqrt{4-5} = -2 \pm i$$

The general solution of the ODE is given by

$$y = A e^{(-2+i)x} + B e^{(-2-i)x}$$

$$= e^{-2x} \left(A e^{ix} + B e^{-ix} \right)$$

$$= e^{-2x} \left[(A+B) \cos x + i(A-B) \sin x \right]$$

After redefinition of the integration constants, $C := A + B$ and $D := i(A-B)$ we can write the general solution as

$$y = e^{-2x} \left(C \cos x + D \sin x \right), \quad C, D \in \mathbb{R}.$$

3.5.3 Inhomogeneous ODE's

We now return to the original inhomogeneous ODE (3.31),

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x).$$

The first step is to obtain a general solution of the corresponding homogeneous ODE (obtained by setting $f(x) = 0$), using the methods discussed in the previous section. This solution is called the *complementary function* $y_{\mathrm{CF}}(x)$. Let us suppose that somehow we can find a particular solution of the inhomogeneous ODE (3.31). Such a solution is called the *particular integral* $y_{\mathrm{PI}}(x)$. Then the sum

$$y(x) = y_{\mathrm{CF}}(x) + y_{\mathrm{PI}}(x) \tag{3.40}$$

is the general solution of the inhomogeneous ODE since

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \underbrace{\left(\frac{d^2y_{\mathrm{CF}}}{dx^2} + p\frac{dy_{\mathrm{CF}}}{dx} + qy_{\mathrm{CF}}\right)}_{=0} + \underbrace{\left(\frac{d^2y_{\mathrm{PI}}}{dx^2} + p\frac{dy_{\mathrm{PI}}}{dx} + qy_{\mathrm{PI}}\right)}_{=f(x)} = f(x),$$

and $y_{\mathrm{CF}}(x)$ contains two arbitrary integration constants. This means that the whole problem is solved if we have a way of finding $y_{\mathrm{PI}}(x)$. Unfortunately, this is usually down to a matter of trial and error.

In these lectures, we will only describe how to find a particular integral $y_{\mathrm{PI}}(x)$ for some important and common types of functions $f(x)$.

- **Polynomials.** If $f(x)$ is an nth degree polynomial,

  $$f(x) = A_0 + A_1x + \ldots + A_nx^n, \tag{3.41}$$

  then there is always a particular integral of the form

  $$y_{\mathrm{PI}}(x) = \alpha_0 + \alpha_1x + \ldots + \alpha_nx^n. \tag{3.42}$$

  To determine the coefficients $\alpha_0, \ldots, \alpha_n$ for any given $p, q, A_0, \ldots A_n$ we insert the ansatz (3.41) into the inhomogeneous ODE,

  $$\begin{aligned} A_0 + A_1x + \ldots + A_nx^n &= 2\alpha_2 + 6\alpha_3x + \ldots + \alpha_nn(n-1)x^{n-2} \\ &\quad +p\left(\alpha_1 + 2\alpha_2x + \ldots + n\alpha_nx^{n-1}\right) \\ &\quad +q\left(\alpha_0 + \alpha_1x + \ldots + \alpha_nx^n\right) \\ &= (2\alpha_2 + p\alpha_1 + q\alpha_0) + (6\alpha_3 + 2p\alpha_2 + q\alpha_1)\,x \\ &\quad + \ldots + q\alpha_nx^n. \end{aligned}$$

  For the two polynomials to be equal, the coefficients have to be equal, leading to $(n+1)$ coupled linear equations,

  $$\begin{aligned} 2\alpha_2 + p\alpha_1 + q\alpha_0 &= A_0 \\ 6\alpha_3 + 2p\alpha_2 + q\alpha_1 &= A_1 \\ &\vdots \\ q\alpha_n &= A_n. \end{aligned}$$

  Solving this set of equations we obtain $\alpha_0, \ldots, \alpha_n$.

- **Exponentials.** If $f(x)$ is an exponential function,

$$f(x) = A_0 e^{\omega x}, \tag{3.43}$$

then the particular integral is

$$y_{\mathrm{PI}}(x) = \alpha_0 e^{\omega x}, \tag{3.44}$$

where α_0 is a constant related to p, q, A_0, and ω. To find a formula for α_0, we insert the ansatz (3.44) into the inhomogeneous ODE,

$$\begin{aligned} A_0 e^{\omega x} &= \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy \\ &= \alpha_0 \omega^2 e^{\omega x} + p\alpha_0 \omega e^{\omega x} + q\alpha_0 e^{\omega x} \\ &= \alpha_0 \left(\omega^2 + p\omega + q\right) e^{\omega x}. \end{aligned} \tag{3.45}$$

From this it follows that

$$\alpha_0 = \frac{A_0}{\omega^2 + p\omega + q}. \tag{3.46}$$

Note that if ω is equal to one of the roots $k_{1,2}$ of the quadratic equation $k^2+pk+q = 0$, then the bracket in Eq. (3.45) is zero and our ansatz does not solve the inhomogeneous ODE. In this case the particular integral has the form

$$y_{\mathrm{PI}}(x) = Bxe^{\omega x}. \tag{3.47}$$

To check this and to determine B, we insert the ansatz into the ODE,

$$\begin{aligned} A_0 e^{\omega x} &= \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy \\ &= \left(\frac{d}{dx} + p\right)\left(Be^{\omega x} + B\omega x e^{\omega x}\right) + qBxe^{\omega x} \\ &= 2B\omega e^{\omega x} + B\omega^2 x e^{\omega x} + pBe^{\omega x} + pB\omega x e^{\omega x} + qBxe^{\omega x} \\ &= Be^{\omega x}\left(2\omega + \omega^2 x + p + p\omega x + qx\right) \\ &= Be^{\omega x}[(2\omega + p) + \underbrace{(\omega^2 + p\omega + q)}_{=0} x]. \end{aligned}$$

If $2\omega + p \neq 0$, then our ansatz (3.47) solves the inhomogeneous ODE for $B = A_0/(2\omega + p)$. In the very special case that the roots $k_{1,2}$ are degenerate and equal to ω, we have $k_1 = k_2 = -p/2 = \omega$. In this case the particular integral is given by $y_{\mathrm{PI}}(x) = Cx^2 e^{\omega x}$ with $C = A_0/2$.

- **Cosine and sine functions.** If $f(x)$ is a periodic function of the form

$$f(x) = A_0 \cos(\omega x) + A_1 \sin(\omega x), \tag{3.48}$$

the the particular integral is

$$y_{\mathrm{PI}}(x) = \alpha_0 \cos(\omega x) + \alpha_1 \sin(\omega x), \tag{3.49}$$

with coefficients α_0 and α_1 expressible in terms of p, q, A_0, A_1 and ω. To see this, note that

$$\begin{aligned}\frac{dy_{\mathrm{PI}}}{dx} &= -\alpha_0\omega\sin(\omega x)+\alpha_1\omega\cos(\omega x),\\ \frac{d^2y_{\mathrm{PI}}}{dx^2} &= -\alpha_0\omega^2\cos(\omega x)-\alpha_1\omega^2\sin(\omega x).\end{aligned}$$

Inserting into the inhomogeneous ODE, we obtain

$$\begin{aligned}A_0\cos(\omega x)+A_1\sin(\omega x) &= \big(-\alpha_0\omega^2+p\alpha_1\omega+q\alpha_0\big)\cos(\omega x)\\ &\quad+\big(-\alpha_1\omega^2-p\alpha_0\omega+q\alpha_1\big)\sin(\omega x).\end{aligned}$$

Requiring the cosine and sine terms to be equal separately, we have

$$\begin{aligned}(q-\omega^2)\alpha_0+p\omega\alpha_1 &= A_0,\\ -p\omega\alpha_0+(q-\omega^2)\alpha_1 &= A_1.\end{aligned}$$

These are simultaneous linear equations which can be solved to obtain α_0 and α_1.

3.5.4 Worked Examples

Find the solutions of the inhomogeneous second-order ODE's

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = f(x),$$

with

$$\text{(a)}\ \ f(x) = 16x + 12, \quad \text{(b)}\ \ f(x) = 5\cos x,$$

for which $y = 1$ and $dy/dx = 0$ at $x = 0$.

We first determine the complimentary function $y_{CF}(x)$, given by the general solution of the homogeneous ODE $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$.

Inserting the ansatz $y(x) = e^{kx}$ we obtain

$$k^2 e^{kx} - 6k e^{kx} + 8 e^{kx} = 0 \Rightarrow k^2 - 6k + 8 = 0$$

$$\Rightarrow k_{1,2} = 3 \pm \sqrt{9-8} = 3 \pm 1 \Rightarrow k_1 = 2 \text{ and } k_2 = 4$$

$$\rightarrow y_{CF}(x) = Ae^{2x} + Be^{4x}, \quad A, B \in \mathbb{R}.$$

(a) For the particular integral, we make the ansatz $y_{PI}(x) = \alpha_0 + \alpha_1 x$. Inserting into the inhomogeneous ODE,

$$-6\alpha_1 + 8(\alpha_0 + \alpha_1 x) = 16x + 12$$

$$\Rightarrow \begin{cases} 8\alpha_1 = 16 \\ 8\alpha_0 - 6\alpha_1 = 12 \end{cases} \Rightarrow \alpha_1 = 2 \text{ and } \alpha_0 = 3$$

$$\rightarrow y_{PI}(x) = 2x + 3.$$

$$y(x) = y_{CF}(x) + y_{PI}(x) = Ae^{2x} + Be^{4x} + 2x + 3$$

Use initial conditions to determine A, B:

$$\left.\frac{dy}{dx}\right|_{x=0} = 2A + 4B + 2 = 0 \Leftrightarrow A + 2B = -1 \ \text{(I)}$$

$$y(0) = A + B + 3 = 1 \Leftrightarrow A + B = -2 \ \text{(II)}$$

I−II: $B = 1$

→II $A = -3$ $\quad \rightarrow y(x) = -3e^{2x} + e^{4x} + 2x + 3$

(b) For the particular integral we make the ansatz

$$y_{PI}(x) = \alpha_0 \cos x + \alpha_1 \sin x .$$

⇝ inhomogeneous ODE:

$$-\alpha_0 \cos x - \alpha_1 \sin x - 6(-\alpha_0 \sin x + \alpha_1 \cos x)$$
$$+8(\alpha_0 \cos x + \alpha_1 \sin x)$$
$$= (-\alpha_0 - 6\alpha_1 + 8\alpha_0) \cos x + (-\alpha_1 + 6\alpha_0 + 8\alpha_1)$$
$$\overset{!}{=} 5 \cos x$$

$$\Rightarrow \begin{cases} 7\alpha_0 - 6\alpha_1 = 5 & (\mathrm{I}) \\ 6\alpha_0 + 7\alpha_1 = 0 & (\mathrm{II}) \end{cases}$$

$7 \cdot \mathrm{I} + 6 \cdot \mathrm{II}$: $\quad 85\alpha_0 = 35 \Rightarrow \alpha_0 = \frac{7}{17}$

⇝ II: $\frac{42}{17} + 7\alpha_1 = 0 \Rightarrow \alpha_1 = -\frac{1}{7} \cdot \frac{42}{17} = -\frac{6}{17}$

$$\rightsquigarrow y_{PI}(x) = \frac{7}{17} \cos x - \frac{6}{17} \sin x$$

$$\rightsquigarrow y(x) = y_{CF}(x) + y_{PI}(x)$$
$$= Ae^{2x} + Be^{4x} + \frac{7}{17} \cos x - \frac{6}{17} \sin x$$

Use initial conditions to determine A and B:

$$\left.\frac{dy}{dx}\right|_{x=0} = 2A + 4B - \frac{6}{17} = 0 \Leftrightarrow A + 2B = \frac{3}{17} \quad (\mathrm{I})$$
$$y(0) = A + B + \frac{7}{17} = 1 \Leftrightarrow A + B = \frac{10}{17} \quad (\mathrm{II})$$

I−II: $B = -\frac{7}{17}$

⇝ II. $A = \frac{17}{17} = 1 \quad \rightsquigarrow y(x) = e^{2x} - \frac{7}{17}e^{4x} + \frac{7}{17}\cos x - \frac{6}{17}\sin x$