INTRODUCTION TO MATRICES

When we wish to solve large systems of simultaneous linear equations, which arise for example in the problem of finding the forces on members of a large framed structure, we can isolate the coefficients of the variables as a block of numbers called a matrix. There are many other applications matrices. In this Section we develop the terminology and basic properties of a matrix.

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Representing simultaneous linear equations

Suppose that we wish to solve the following three equations in three unknowns x_1, x_2 and x_3:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 3 \\ x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 + 4x_3 &= 5 \end{aligned}$$

We can isolate three facets of this system: the **coefficients** of x_1, x_2, x_3; the **unknowns** x_1, x_2, x_3; and the **numbers** on the right-hand sides.
Notice that in the system

$$\begin{aligned} 3x + 2y - z &= 3 \\ x - y + z &= 4 \\ 2x + 3y + 4z &= 5 \end{aligned}$$

the only difference from the first system is the names given to the unknowns. It can be checked that the first system has the solution $x_1 = 2,\ x_2 = -1,\ x_3 = 1$. The second system therefore has the solution $x = 2, y = -1, z = 1$.
We can isolate the three facets of the first system by using **arrays** of numbers and of unknowns:

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Even more conveniently we represent the arrays with letters (usually capital letters)

$$AX = B$$

Here, to be explicit, we write

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Here A is called the **matrix of coefficients**, X is called the **matrix of unknowns** and B is called the **matrix of constants**.
If we now append to A the column of right-hand sides we obtain the **augmented matrix** for the system:

Definitions:

An array of numbers, rectangular in shape, is called a matrix. The first matrix below has 3 rows and 2 columns and is said to be a '3 by 2' matrix (written 3 × 2). The second matrix is a '2 by 4' matrix (written 2 × 4).

$$\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$$

The general 3×3 matrix can be written

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where a_{ij} denotes the element in row i, column j.
For example in the matrix:

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 6 & -12 \\ 5 & 7 & 123 \end{bmatrix}$$

$$a_{11} = 0, \quad a_{12} = -1, \quad a_{13} = -3, \quad \ldots \quad a_{22} = 6, \quad \ldots \quad a_{32} = 7, \quad a_{33} = 123$$

COLUMN MATRIX

A matrix with only one column is called a **column vector** (or **column matrix**).

For example, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ are both 3×1 column vectors.

RAW MATRIX:

A matrix with only one row is called a row vector (or row matrix). For example [2, −3, 8, 9] is a 1 × 4 row vector. Often the entries in a row vector are separated by commas for clarity.

Square matrix:

When the number of rows is the same as the number of columns, i.e. $m = n$, the matrix is said to be **square** and of **order** n (or m).

- In an $n \times n$ square matrix A, the **leading diagonal** (or **principal diagonal**) is the 'north-west to south-east' collection of elements $a_{11}, a_{22}, \dots, a_{nn}$. The sum of the elements in the leading diagonal of A is called the **trace** of the matrix, denoted by tr(A).

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

- A square matrix in which all the elements below the leading diagonal are zero is called an **upper triangular matrix**, often denoted by U.

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & \dots & u_{1n} \\ 0 & u_{22} & \dots & \dots & u_{2n} \\ 0 & 0 & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix} \qquad u_{ij} = 0 \quad \text{when} \quad i > j$$

- A square matrix in which all the elements above the leading diagonal are zero is called a **lower triangular matrix**, often denoted by L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & 0 \\ l_{n1} & l_{n2} & \vdots & \dots & l_{nn} \end{bmatrix} \qquad l_{ij} = 0 \quad \text{when} \quad i < j$$

- A square matrix where all the non-zero elements are along the leading diagonal is called a diagonal matrix, often denoted by D.

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & d_{nn} \end{bmatrix} \qquad d_{ij} = 0 \quad \text{when} \quad i \neq j$$

Some examples of matrices and their classification:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is } 2 \times 3. \text{ It is not square.}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ is } 2 \times 2. \text{ It is square.}$$

Also, tr(A) does not exist, and tr$(B) = 1 + 4 = 5$.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix} \text{ are both } 3 \times 3, \text{ square and upper triangular.}$$

Also, tr$(C) = 0$ and tr$(D) = 3$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 1 \end{bmatrix} \text{ and } F = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ are both } 3 \times 3, \text{ square and lower triangular.}$$

Also, tr$(E) = 0$ and tr$(F) = 4$.

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ are both } 3 \times 3, \text{ square and diagonal.}$$

Also, tr$(G) = 0$ and tr$(H) = 6$.

EXAMPLE NO.1:

Classify the following matrices (and, where possible, find the trace):

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -3 & -2 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

SOLUTION:

A is 3×2, B is 3×4, C is 4×4 and square.

The trace is not defined for A or B. However, tr$(C) = 34$.

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EXAMPLE NO.2:

Classify the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer
A is 3×3 and square, B is 3×3 lower triangular, C is 3×3 upper triangular and D is 3×3 diagonal.

EQUALITY OF MATRICES:

As we noted earlier, the terms in a matrix are called the **elements** of the matrix.

The elements of the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$ are $1, 2, -1, -4$

We say two matrices A, B are **equal** to each other only if A and B have the same number of rows and the same number of columns and if each element of A is equal to the corresponding element of B. When this is the case we write $A = B$. For example if the following two matrices are equal:

$$A = \begin{bmatrix} 1 & \alpha \\ -1 & -\beta \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

then we can conclude that $\alpha = 2$ and $\beta = 4$.

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UNIT MATRIX:

The **unit matrix** or the **identity matrix**, denoted by I_n (or, often, simply I), is the diagonal matrix of order n in which all diagonal elements are 1.

Hence, for example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

ZERO MATRIX:

The **zero matrix** or **null matrix** is the matrix all of whose elements are zero. There is a zero matrix for every size. For example the 2×3 and 2×2 cases are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Zero matrices, of whatever size, are denoted by $\underline{0}$.

TRANSPOSE MATRIX:

The **transpose** of a matrix A is a matrix where the rows of A become the columns of the new matrix and the columns of A become its rows. For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The resulting matrix is called the **transposed matrix** of A and denoted A^T. In the previous example it is clear that A^T is not equal to A since the matrices are of different sizes. If A is square $n \times n$ then A^T will also be $n \times n$.

EXAMPLE NO.3:

Find the transpose of the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution

Interchanging rows with columns we find

$$B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Both matrices are 3×3 but B and B^T are clearly different.

When the transpose of a matrix is equal to the original matrix i.e. AT = A, then we say that the matrix A is symmetric. (This is because it has symmetry about the leading diagonal.) In above Example B is not symmetric.

EXAMPLE NO. 4:

Show that the matrix $C = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

Solution

Taking the transpose of C:

$$C^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}.$$

Clearly $C^T = C$ and so C is a symmetric matrix. Notice how the leading diagonal acts as a "mirror"; for example $c_{12} = -2$ and $c_{21} = -2$. In general $c_{ij} = c_{ji}$ for a symmetric matrix.

EXERCISE:

Find the transpose of each of the following matrices. Which are symmetric?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = C, \text{ symmetric}$$

$$D^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \quad E^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E, \text{ symmetric}$$

Addition and subtraction of matrices:

Under what circumstances can we add two matrices i.e. define $A + B$ for given matrices A, B?

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 & 9 \\ 7 & 8 & 10 \end{bmatrix}$$

There is no sensible way to define $A + B$ in this case since A and B are different sizes.
However, if we consider matrices of the same size then addition can be defined in a very natural way. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. The 'natural' way to add A and B is to add corresponding elements together:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

In general if A and B are both $m \times n$ matrices, with elements a_{ij} and b_{ij} respectively, then their sum is a matrix C, also $m \times n$, such that the elements of C are

$$c_{ij} = a_{ij} + b_{ij} \qquad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n$$

In the above example

$$c_{11} = a_{11} + b_{11} = 1 + 5 = 6 \qquad c_{21} = a_{21} + b_{21} = 3 + 7 = 10 \quad \text{and so on.}$$

Subtraction of matrices follows along similar lines:

$$D = A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Multiplication of a matrix by a number:

There is also a natural way of defining the product of a matrix with a number. Using the matrix A above, we note that

$$A + A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

What we see is that $2A$ (which is the shorthand notation for $A + A$) is obtained by multiplying *every* element of A by 2.
In general if A is an $m \times n$ matrix with typical element a_{ij} then the product of a number k with A is written kA and has the corresponding elements ka_{ij}.

Hence, again using the matrix A above,

$$7A = 7\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$$

Similarly:

$$-3A = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix}$$

EXCERCISE:

For the following matrices find, where possible, $A + B,\ A - B,\ B - A,\ 2A$.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Answer

1. $A+B=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $A-B=\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ $B-A=\begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix}$ $2A=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

2. $A+B=\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 8 & 9 & 10 \end{bmatrix}$ $A-B=\begin{bmatrix} 0 & 1 & 2 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix}$ $B-A=\begin{bmatrix} 0 & -1 & -2 \\ -5 & -6 & -7 \\ -6 & -7 & -8 \end{bmatrix}$

   $2A=\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

3. None of $A+B,\ A-B,\ B-A,$ are defined. $2A=\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

Some simple matrix properties:

Matrix addition is commutative: A + B = B + A

Matrix addition is associative: A + (B + C) = (A + B) + C

The distributive law holds: k(A + B) = k A + k B

$(A + B)^T = A^T + B^T$

$(A - B)^T = A^T - B^T$

$(A^T)^T = A$

EXAMPLE NO.5:

Show that $(A^T)^T = A$ for the matrix $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Solution

$A^T=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ so that $(A^T)^T=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}=A$

EXAMPLE NO.6:

For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $3(A+B) = 3A + 3B$ (ii) $(A-B)^T = A^T - B^T$.

Answer

(i) $A + B = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$; $3(A+B) = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$; $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$;

$3B = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$; $3A + 3B = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$.

(ii) $A - B = \begin{bmatrix} 0 & 3 \\ 4 & 3 \end{bmatrix}$; $(A-B)^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$;

$B^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $A^T - B^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$.

Exercises

1. Find the coefficient matrix A of the system:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ 4x_1 + 4x_2 &= 0 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

   If $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ determine $(3A^T - B)^T$.

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$ verify that $3(A^T - B) = (3A - 3B^T)^T$.

Answers

1. $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & 0 \\ 2 & -1 & -1 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 4 & -1 \\ -1 & 0 & -1 \end{bmatrix}$, $3A^T = \begin{bmatrix} 6 & 12 & 6 \\ 9 & 12 & -3 \\ -3 & 0 & -3 \end{bmatrix}$

   $3A^T - B = \begin{bmatrix} 5 & 10 & 3 \\ 5 & 7 & -9 \\ -3 & 0 & -4 \end{bmatrix}$ $(3A^T - B)^T = \begin{bmatrix} 5 & 5 & -3 \\ 10 & 7 & 0 \\ 3 & -9 & -4 \end{bmatrix}$

2. $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $A^T - B = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $3(A^T - B) = \begin{bmatrix} 6 & 0 \\ 6 & 12 \\ 3 & -3 \end{bmatrix}$

   $B^T = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & 7 \end{bmatrix}$, $3A - 3B^T = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 6 \\ 12 & 3 & 21 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 0 & 12 & -3 \end{bmatrix}$