# 1. Modelling with first-order equations

## Applying Newton's law of cooling

In Section 19.1 we introduced Newton's law of cooling. The model equation is

$$\frac{d\theta}{dt} = -k(\theta - \theta_{\mathsf{s}}) \qquad \theta = \theta_0 \text{ at } t = 0. \qquad (5)$$

where $\theta = \theta(t)$ is the temperature of the cooling object at time $t$, $\theta_{\mathsf{s}}$ the temperature of the environment (assumed constant) and $k$ is a thermal constant related to the object, $\theta_0$ is the initial temperature of the liquid.

**Task**

Solve this initial value problem:

$$\frac{d\theta}{dt} = -k(\theta - \theta_s), \qquad \theta = \theta_0 \quad \text{at} \quad t = 0$$

Separate the variables to obtain an equation connecting two integrals:

**Your solution**

**Answer**

$$\int \frac{d\theta}{\theta - \theta_{\mathsf{s}}} = -\int k\ dt$$

Now integrate both sides of this equation:

**Your solution**

**Answer**

$\ln(\theta - \theta_{\mathsf{s}}) = -kt + C$ where $C$ is constant

Apply the initial condition and take exponentials to obtain a formula for $\theta$:

**Your solution**

**Answer**

$\ln(\theta_0 - \theta_{\mathsf{s}}) = C$. Hence $\ln(\theta - \theta_{\mathsf{s}}) = -kt + \ln(\theta_0 - \theta_{\mathsf{s}})$ so that $\ln(\theta - \theta_{\mathsf{s}}) - \ln(\theta_0 - \theta_0) = -kt$

Thus, rearranging and inverting, we find:

$$\ln\left(\frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}}\right) = -kt \qquad \therefore \qquad \frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}} = \mathrm{e}^{-kt} \quad \text{giving} \quad \theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathrm{e}^{-kt}.$$

The graph of $\theta$ against $t$ for $\theta = \theta_s + (\theta_0 - \theta_s)\mathrm{e}^{-kt}$ is shown in Figure 4 below.

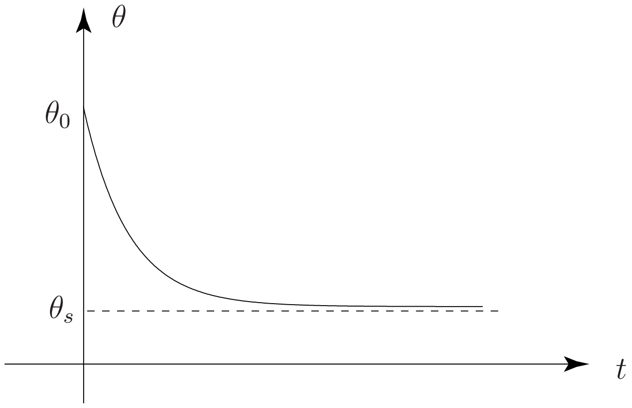


**Figure 4**

We see that as time increases ($t \to \infty$), then the temperature of the object cools down to that of the environment, that is: $\theta \to \theta_{\mathsf{s}}$.

We could have solved (5) by the integrating factor method, which you are now asked to do.

**Task**

We can write the equation for Newton's law of cooling (5) as

$$\frac{d\theta}{dt} + k\,\theta = k\theta_{\mathsf{s}}, \qquad \theta = \theta_0 \text{ at } t = 0 \qquad (6)$$

State the integrating factor for this equation:

**Your solution**

**Answer**

$\mathrm{e}^{\int k\,dt} = \mathrm{e}^{kt}$ is the integrating factor.

Multiplying (6) by this factor we find that

$$\mathrm{e}^{kt}\frac{d\theta}{dt} + k\mathrm{e}^{kt}\theta = k\theta_{\mathsf{s}}\mathrm{e}^{kt} \qquad \text{or, rearranging,} \qquad \frac{d}{dt}(\mathrm{e}^{kt}\theta) = k\theta_{\mathsf{s}}\mathrm{e}^{kt}$$

Now integrate this equation and apply the initial condition:

**Your solution**

**Answer**

Integration produces $\mathrm{e}^{kt}\theta = \theta_{\mathsf{s}}\mathrm{e}^{kt} + C$, where $C$ is an arbitrary constant. Then, applying the initial condition: when $t = 0$, $\theta_0 = \theta_{\mathsf{s}} + C$ so that $C = \theta_0 - \theta_{\mathsf{s}}$ gives the same result as before:

$$\theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathrm{e}^{-kt},$$

## Modelling electrical circuits

Another application of first-order differential equations arises in the modelling of electrical circuits. In Section 19.1 the differential equation for the RL circuit in Figure 5 below was shown to be

$$L\,\frac{di}{dt} + Ri = E$$

in which the initial condition is $i = 0$ at $t = 0$.

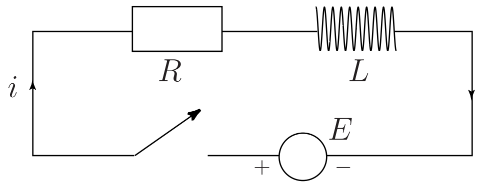


**Figure 5**

First we write this equation in standard form $\{\frac{dy}{dx} + P(x)y = Q(x)\}$ and obtain the integrating factor.

Dividing the differential equation through by $L$ gives

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}$$

which is now in standard form. The integrating factor is $\mathrm{e}^{\int \frac{R}{L}dt} = \mathrm{e}^{Rt/L}$.
Multiplying the equation in standard form by the integrating factor gives

$$\mathrm{e}^{Rt/L}\frac{di}{dt} + \mathrm{e}^{Rt/L}\frac{R}{L}i = \frac{E}{L}\mathrm{e}^{Rt/L}$$

or, rearranging,

$$\frac{d}{dt}(\mathrm{e}^{Rt/L}\;i) = \frac{E}{L}\mathrm{e}^{Rt/L}.$$

Now we integrate both sides and apply the initial condition to obtain the solution.

Integrating the differential equation gives:

$$\mathrm{e}^{Rt/L}\;i = \frac{E}{R}\;\mathrm{e}^{Rt/L} + C$$

where $C$ is a constant so

$$i = \frac{E}{R} + C\mathrm{e}^{-Rt/L}$$

Applying the initial condition $i = 0$ when $t = 0$ gives

$$0 = \frac{E}{R} + C$$

so that $C = -\dfrac{E}{R}$.

Finally, $i = \dfrac{E}{R}(1 - \mathrm{e}^{-Rt/L})$.

Note that as $t \to \infty$, $i \to \dfrac{E}{R}$ so as $t$ increases the effect of the inductor diminishes to zero.