Ordinary Differential Equations

1 Introduction

A **differential equation** is an equation relating an *independent* variable, e.g. t, a *dependent* variable, y, and one or more derivatives of y with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x \qquad y^2\frac{\mathrm{d}y}{\mathrm{d}t} = e^t \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In this section we will look at some specific types of differential equation and how to solve them.

2 Classifying equations

We can classify our differential equation by four properties:

- Is it an **ordinary** differential equation?
- Is it **linear**?
- Does it have **constant coefficients**?
- What is the **order**?

Ordinary

An Ordinary Differential Equation or ODE has only one independent variable (for example, x, or t). The alternative (with more than one) is called a partial differential equation and will not be covered in this course.

Linearity

A differential equation is linear if every term in the equation contains none or exactly one of either the dependent variable or its derivatives. There are no products of the dependent variable with itself or its derivatives. Each term has at most one power of the equivalent of x or $\dot{x}$ or $\ddot{x}$ or $\ldots$; or $f(x)$ and its derivatives.

Examples:

$$f(x)\frac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2x \text{ is not linear} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = f^3(x) \text{ is not linear} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = -x^2f(x) + e^x \text{ is linear.}$$

Constant coefficients

A differential equation has constant coefficients if the dependent variable and all the derivatives are only multiplied by constants.

Examples: which have constant coefficients?

$$3\frac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2x\text{: yes} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = -x^2f(x) + e^x\text{: no} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} + 3\frac{\mathrm{d}f}{\mathrm{d}x} + 2f(x) = \sin xe^x\text{: no.}$$

Finally, a "trick" one:

$$3e^x \frac{\mathrm{d}f}{\mathrm{d}x} + e^x f(x) = x^3 \text{ **does** have constant coefficients: divide the whole equation by } e^x.$$

Order

The <u>order</u> of a differential equation is the largest number of derivatives (of the dependent variable) ever taken.

Examples:

$f(x)\frac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2 x$ is 1st order $\quad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -x^2 f(x) + e^x$ is 2nd order $\quad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ is 2nd order.

3 First order linear equations

First the general theory. A first order linear differential equation for $y(x)$ must be of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

If there is something multiplying the $\mathrm{d}y/\mathrm{d}x$ term, then divide the whole equation by this first.

Now suppose we calculate an **integrating factor**

$$I(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

Just this once, we won't bother about a constant of integration.

We multiply our equation by the integrating factor:

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)p(x)y = I(x)q(x).$$

and then observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + y\frac{\mathrm{d}I}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + yp(x)I(x)$$

which is our left-hand-side. So we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = I(x)q(x)$$

which we can integrate (we hope):

$$yI(x) = \int I(x)q(x)\,\mathrm{d}x + C$$

$$y = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x + \frac{C}{I(x)}.$$

We sort out the constant C from the initial conditions **at the end**.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 0 \qquad \text{and} \qquad y = 3 \text{ when } x = 0.$$

Here the integrating factor will be

$$I(x) = \exp\left(\int 2x \,\mathrm{d}x\right) = \exp x^2$$

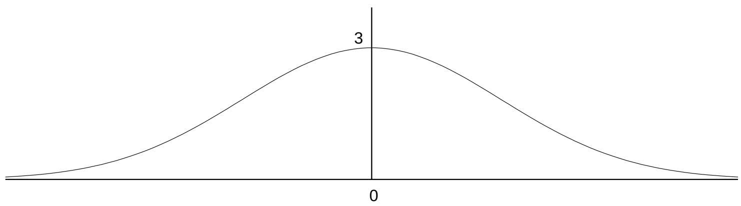
and our equation is

$$e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y = 0.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[ye^{x^2}\right] = 0 \qquad \Longrightarrow \qquad ye^{x^2} = C \qquad \Longrightarrow \qquad y = Ce^{-x^2}.$$

The last thing we do is use the initial conditions: at $x = 0$, $y = 3$ but our form gives at $x = 0$, $y = C$ so we need $C = 3$ and

$$y = 3e^{-x^2}.$$



Example

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin x \qquad \text{with } y(\pi/2) = 0.$$

First we need to get the equation into a form where the first term is just $\mathrm{d}y/\mathrm{d}x$: so divide by x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\sin x}{x}.$$

Now we calculate the integrating factor:

$$I(x) = \exp\left(\int \frac{2}{x}\,\mathrm{d}x\right) = \exp(2\ln x) = \exp\ln(x^2) = x^2.$$

We multiply the whole system by x^2:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\sin x$$

and now we can integrate:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = x\sin x \qquad \Longrightarrow \qquad x^2y = \int x\sin x\,\mathrm{d}x + C$$

which we can integrate by parts:

$$x^2 y = -x\cos x + \int \cos x \,\mathrm{d}x + C = -x\cos x + \sin x + C$$

so the general solution is

$$y = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}.$$

Finally, we use the initial condition $y = 0$ when $x = \pi/2$ to get

$$0 = -\frac{\cos(\pi/2)}{(\pi/2)} + \frac{\sin(\pi/2)}{(\pi/2)^2} + \frac{C}{(\pi/2)^2} = 0 + \frac{1}{(\pi/2)^2} + \frac{C}{(\pi/2)^2}.$$

which means $C = -1$ and our solution is

$$y = -\frac{\cos x}{x} - \frac{1 - \sin x}{x^2}.$$

Example

This time we will solve two different differential equations in parallel.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-2x} \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} + 3f = e^{-3x}$$

In this example, we don't actually have variable coefficients – but that just makes it easier!

$$\text{In both cases,} \quad I(x) = \exp\int 3\,\mathrm{d}x = e^{3x}.$$

$$\begin{aligned}
e^{3x}\frac{\mathrm{d}y}{\mathrm{d}x} + 3e^{3x}y = e^{x} \qquad &\text{and} \qquad e^{3x}\frac{\mathrm{d}f}{\mathrm{d}x} + 3e^{3x}f = 1.\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x}y\right) = e^{x} \qquad &\text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x}f\right) = 1.\\
e^{3x}y = e^{x} + C_0 \qquad &\text{and} \qquad e^{3x}f = x + C_1.\\
y = e^{-2x} + C_0e^{-3x} \qquad &\text{and} \qquad f = xe^{-3x} + C_1e^{-3x}.
\end{aligned}$$

4 Homogeneous linear equations.

A **homogeneous** linear equation is one in which all terms contain **exactly** one power of the dependent variable and its derivatives:

$$\text{e.g.} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

For these equations, we can add up solutions: so if $f(x)$ is a solution and $g(x)$ is a solution:

$$\frac{\mathrm{d}^2f}{\mathrm{d}x^2} + 5\frac{\mathrm{d}f}{\mathrm{d}x} + 6f = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2g}{\mathrm{d}x^2} + 5\frac{\mathrm{d}g}{\mathrm{d}x} + 6g = 0$$

then so is $af(x) + bg(x)$ for any constants a and b:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[af(x) + bg(x)] + 5\frac{\mathrm{d}}{\mathrm{d}x}[af(x) + bg(x)] + 6[af(x) + bg(x)] = 0.$$

An nth order homogeneous linear equation will "always" (i.e. if it is well-behaved: don't worry about this detail) have exactly n independent solutions $y_1, \ldots, y_n$ and the general solution to the equation is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

4.1 Special case: coefficients ax^r

Suppose we are given a differential equation in which the coefficient of the rth derivative is a constant multiple of x^r:

$$\text{e.g. } x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0.$$

Then if we try a solution of the form $y = x^m$ we get

$$y = x^m \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m(m-1)x^{m-2}$$

and if we put this back into the original equation we get

$$x^2 m(m-1)x^{m-2} + 2mxx^{m-1} - 6x^m = 0$$

$$x^m(m(m-1) + 2m - 6) = 0 \qquad x^m(m^2 + m - 6) = 0.$$

Now x^m will take lots of values as x changes so we need

$$(m^2 + m - 6) = 0 \qquad \Longrightarrow \qquad (m-2)(m+3) = 0.$$

In this case we get two roots: $m_1 = 2$ and $m_2 = -3$. This means we have found two functions that work as solutions to our differential equation:

$$y_1 = x^{m_1} = x^2 \qquad \text{and} \qquad y_2 = x^{m_2} = x^{-3}.$$

But we know that if we have two solutions we can use any combination of them so our *general solution* is

$$y = c_1 x^2 + c_2 x^{-3}.$$

This works with an nth order ODE as long as the nth order polynomial for m has n different real roots.

Example

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 0.$$

Try $y = x^m$:

$$y = x^m \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m(m-1)x^{m-2}.$$

$$m(m-1)x^m - 6mx^m + 10x^m = 0 \implies x^m(m^2 - m - 6m + 10) = 0 \implies x^m(m-5)(m-2) = 0.$$

The general solution to this equation is

$$y = c_1 x^5 + c_2 x^2.$$

4.2 Special case: constant coefficients.

Now suppose we have a homogeneous equation with constant coefficients, like this one:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

We try a solution $y = e^{\lambda x}$. This gives $\mathrm{d}y/\mathrm{d}x = \lambda e^{\lambda x}$ and $\mathrm{d}^2y/\mathrm{d}x^2 = \lambda^2 e^{\lambda x}$ so

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 6e^{\lambda x} = 0.$$

$$(\lambda^2 + 5\lambda + 6)e^{\lambda x} = 0 \qquad \text{for all } x.$$

Just like the polynomial case, the function of x will not be zero everywhere so we need

$$\lambda^2 + 5\lambda + 6 = 0 \qquad \Longrightarrow \qquad (\lambda + 2)(\lambda + 3) = 0.$$

In this case we get two roots: $\lambda_1 = -2$ and $\lambda_2 = -3$. This means we have found two independent solutions:

$$y_1 = e^{\lambda_1 x} = e^{-2x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x} = e^{-3x},$$

and the *general solution* is

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

Example

A third-order equation this time:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - \frac{\mathrm{d}^2y}{\mathrm{d}x^3} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^3 - \lambda^2 - 2\lambda = 0 \qquad \Longrightarrow \qquad \lambda(\lambda^2 - \lambda - 2) = 0 \qquad \Longrightarrow \qquad \lambda(\lambda - 2)(\lambda + 1) = 0$$

which has three roots,

$$\lambda_1 = 0 \qquad \lambda_2 = 2 \qquad \lambda_3 = -1.$$

The general solution is

$$y = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-x} = c_1 + c_2 e^{2x} + c_3 e^{-x}.$$

Notice that we have three constants here: in general we will always have N constants in the solution to an Nth order equation.

Example

Another second-order equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0$$

which has two roots,

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i.$$

The general solution is then

$$y = Ae^{(-1+2i)x} + Be^{(-1-2i)x} = e^{-x}[Ae^{2ix} + Be^{-2ix}]$$

where A and B will be complex constants: but if y is real (which it usually is) then we can write the solution as

$$y = e^{-x}[c_1 \sin 2x + c_2 \cos 2x].$$

Repeated roots

If our polynomial for λ has two roots the same, then we will end up one short in our solution. This is similar to the case with a repeated eigenvalue in the previous section: there, we used a generalised eigenvector and a function $xe^{\lambda x}$. Here we only need the $xe^{\lambda x}$ part.

Example

Another third-order equation:

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^3 - 2\lambda^2 + \lambda = 0 \qquad \Longrightarrow \qquad \lambda(\lambda^2 - 2\lambda + 1) = 0 \qquad \Longrightarrow \qquad \lambda(\lambda-1)^2 = 0$$

which has only two distinct roots,

$$\lambda_1 = 0 \qquad \lambda_2 = \lambda_3 = 1.$$

The general solution is

$$y = c_1 e^{0x} + c_2 e^x + c_3 x e^x = c_1 + c_2 e^x + c_3 x e^x.$$

5 Inhomogeneous linear equations.

What happens if there is a term with **none** of the dependent variable? That is, loosely, a term on the right hand side, or a function of x.

$$f_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)y = g(x).$$

In the most general case we can't do anything: but in one or two special cases we can.

If we already know the general solution to the homogeneous equation:

$$f_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)y = 0 \qquad \Longrightarrow \qquad y = c_1 y_1(x) + c_2 y_2(x)$$

then all we need is a particular solution to the whole equation: one function $u(x)$ that obeys

$$f_2(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} + f_0(x)u = g(x).$$

Then the general solution to the whole equation is

$$y = c_1y_1(x) + c_2y_2(x) + u(x).$$

The solution to the homogeneous equation is called the complementary function or CF; the particular solution $u(x)$ is called the particular integral or PI. Finding it involves a certain amount of trial and error!

Special case: Coefficients x^r

In this case, we can only cope with one specific kind of RHS: a polynomial. We will see this by example:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 6x^3.$$

The homogeneous equation in this case is one we've seen before:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 0 \qquad \Longrightarrow \qquad y = c_1x^5 + c_2x^2.$$

Now as long as the power on the right is **not part of the CF** we can find the PI by trying a multiple of the right hand side:

$$y = Ax^3 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 3Ax^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6Ax.$$

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = x^2 6Ax - 6x3Ax^2 + 10Ax^3 = x^3[6A - 18A + 10A] = -2Ax^3$$

so for this to be a solution we need $-2A = 6$ so $A = -3$. Then the general solution to the full equation is

$$y = c_1x^5 + c_2x^2 - 3x^3.$$

A couple of words of warning about this kind of equation:

- If the polynomial for the power m has a repeated root then we fail
- If the polynomial for the power m has complex roots then we fail
- If a power on the RHS matches a power in the CF then we fail.

Special case: constant coefficients

Given a linear ODE with constant coefficients, we saw in the previous section that we can **always** find the general solution to the homogeneous equation (using $e^{\lambda x}$, $xe^{\lambda x}$ and so on), so we know how to find the CF. There are a set of standard functions to try for the PI, but that part is not guaranteed.

Example

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}.$$

First we need the **CF**. Try $y = e^{\lambda x}$ on the homogeneous equation:

$$\lambda^2 - 3\lambda + 2 = 0 \qquad \Longrightarrow \qquad (\lambda-1)(\lambda-2) = 0.$$

So there are two roots, $\lambda_1 = 1$ and $\lambda_2 = 2$. The CF is then

$$y_{\mathrm{CF}} = c_1 e^x + x_2 e^{2x}.$$

Next we need the **PI**. Since the RHS is e^{-x}, we try the form

$$y = Ae^{-x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -Ae^{-x} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Ae^{-x}.$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = Ae^{-x} + 3Ae^{-x} + 2Ae^{-x} = 6Ae^{-x}$$

so we need $A = 1/6$ for this to work. Our PI is

$$y_{\mathrm{PI}} = \frac{1}{6}e^{-x}$$

and the general solution is

$$y = c_1 e^x + x_2 e^{2x} + \frac{1}{6}e^{-x}.$$

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-3x}.$$

This is only first-order: in fact we solved it in section 3 and the solution was

$$y = xe^{-3x} + C_1 e^{-3x}.$$

Let us solve it the way we have just learned. First the CF: try $y = e^{\lambda x}$ then

$$\lambda + 3 = 0$$

so $\lambda = -3$ and the CF is

$$y_{\mathrm{CF}} = C_1 e^{-3x}.$$

Now look for the PI. The RHS is e^{-3x} so our first thought might be to try Ae^{-3x}. But this is the CF: so we know when we try it we will get zero! So instead (motivated by knowing the answer in this case) we multiply by x and try

$$y = Axe^{-3x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = Ae^{-3x} - 3Axe^{-3x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = Ae^{-3x} - 3Axe^{-3x} + 3Axe^{-3x} = Ae^{-3x}.$$

so we need $A = 1$ and we end up with the same solution we got last time.

In general, if the RHS matches the CF (or part of the CF) then we will multiply by x to get our trial function for the PI.

Example

This time we have initial conditions as well: remember we **always** use these as the very last thing we do.

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \qquad \text{with } y = 3,\ \frac{\mathrm{d}y}{\mathrm{d}x} = -4 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4 \text{ at } x = 0.$$

First we find the CF. Try $y = e^{\lambda x}$:

$$\lambda^3 + 2\lambda^2 + \lambda = 0 \implies \lambda(\lambda^2 + 2\lambda + 1) = 0 \implies \lambda(\lambda+1)^2 = 0.$$

This has only two distinct roots: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = -1$. Therefore the CF is:

$$y_{\text{CF}} = c_1 + c_2e^{-x} + c_3xe^{-x}.$$

Now we look for the PI. The RHS is x so we try a function

$$y = Ax + B \implies \frac{\mathrm{d}y}{\mathrm{d}x} = A \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0.$$

This makes

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 + 0 + A$$

and no value of A can make this equal to x. What do we do when it fails?

- If the trial function fails, try multiplying by x.

[Note: in this case we could have predicted this because the B of our trial function is part of the CF.]

We want one more power of x so we try

$$y = Cx^2 + Ax \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + A \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C \text{ and } \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0.$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 + 4C + 2Cx + A$$

so we need

$$2Cx + 4C + A = 2x \qquad \text{which means } C = 1,\ A = -4.$$

Our general solution is

$$y = c_1 + c_2e^{-x} + c_3xe^{-x} + x^2 - 4x.$$

Now we apply the initial conditions:

$$\begin{aligned}
y = c_1 + c_2e^{-x} + c_3xe^{-x} + x^2 - 4x \quad &\implies \quad y(0) = c_1 + c_2 = 3\\
\frac{\mathrm{d}y}{\mathrm{d}x} = -c_2e^{-x} + c_3e^{-x} - c_3xe^{-x} + 2x - 4 \quad &\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = -c_2 + c_3 - 4 = -4\\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = c_2e^{-x} - 2c_3e^{-x} + c_3xe^{-x} + 2 \quad &\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(0) = c_2 - 2c_3 + 2 = 4
\end{aligned}$$

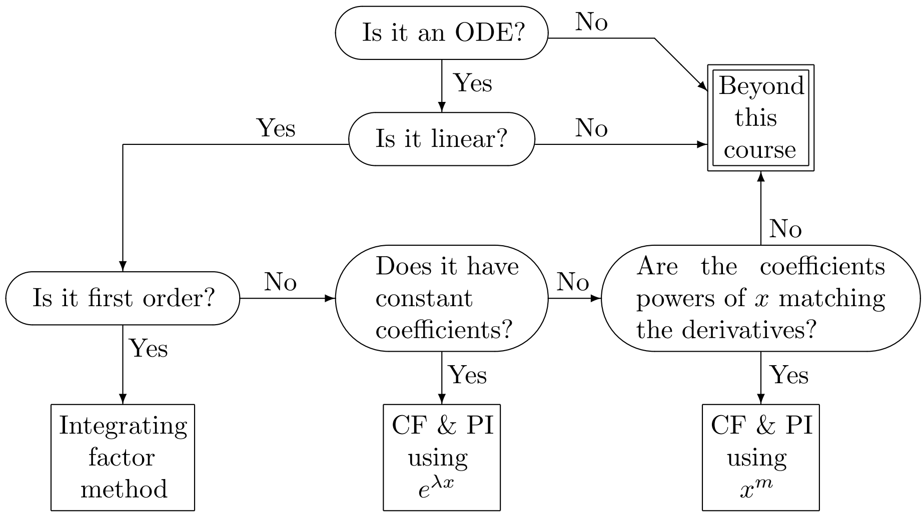
The solution to this linear system is $c_2 = -2$, $c_3 = -2$, $c_1 = 5$ so our final answer is

$$y = 5 - 2e^{-x} - 2xe^{-x} + x^2 - 4x.$$

Table of functions to try for PI

$f(x)$	Conditions on CF	First guess at PI
αe^{kx}	$\lambda = k$ not a root	$Ae^{\lambda x}$
αe^{kx}	$\lambda = k$ a root	$Axe^{\lambda x}$
αe^{kx}	$\lambda = k$ a double root	$Ax^2 e^{\lambda x}$
$\sin kx$	$\lambda = ik$ not a root	$A\cos kx + B\sin kx$
$\cos kx$	$\lambda = ik$ not a root	$A\cos kx + B\sin kx$
$\sin kx$	$\lambda = ik$ a root	$Ax\cos kx + Bx\sin kx$
$\sin kx$	$\lambda = ik$ a double root	$Ax^2\cos kx + Bx^2\sin kx$
x^n	$\lambda = 0$ not a root	$Ax^n + Bx^{n-1} + \cdots + C$
x^n	$\lambda = 0$ a root	$Ax^{n+1} + Bx^n + \cdots + Cx$
x^n	$\lambda = 0$ a double root	$Ax^{n+2} + Bx^{n+1} + \cdots + Cx^2$

6 Summary of differential equations



First-order linear ODEs

To solve an equation of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$, we calculate an integrating factor $I(x) = e^{\int p(x)\,\mathrm{d}x}$ and multiply by it: $\dfrac{\mathrm{d}y}{\mathrm{d}x}e^{\int p\,\mathrm{d}x} + p(x)ye^{\int p\,\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\int p\,\mathrm{d}x}\right) = q(x)e^{\int p\,\mathrm{d}x}$.

We integrate both sides: $ye^{\int p\,\mathrm{d}x} = \displaystyle\int q(x)e^{\int p\,\mathrm{d}x}\,\mathrm{d}x + C \implies y = e^{-\int p\,\mathrm{d}x}\int q(x)e^{\int p\,\mathrm{d}x}\,\mathrm{d}x + Ce^{-\int p\,\mathrm{d}x}$.

Linear ODEs with constant coefficients

To solve an equation of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x)$ we calculate the CF, by trying solutions of the form $y = e^{\lambda x}$ for the homogeneous equation (RHS=0). If there is a repeated root we use $e^{\lambda x}$ and $xe^{\lambda x}$:

$$y_{\mathrm{CF}} = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} \qquad \text{or} \qquad y_{\mathrm{CF}} = c_1e^{\lambda x} + c_2xe^{\lambda x}.$$

We calculate the PI, which is **any** solution to the original equation, by a system of trial and error. In general we try something of the same form as $f(x)$; if this overlaps with the CF at all then we multiply by x. The general solution is $y = y_{\mathrm{CF}} + y_{\mathrm{PI}}$.

Linear ODEs with x^n-type coefficients

To solve an equation of the form $ax^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = x^n$ we calculate the CF, by trying solutions of the form $y = x^m$ for the homogeneous equation. As long as the roots m are *real*, *different* and *not equal to* n this is OK. Then we use $y = Ax^n$ as the PI.