Parial derivative

0.1 Recall: ordinary derivatives

If y is a function of x then $\frac{dy}{dx}$ is the **derivative** meaning the gradient (slope of the graph) or the rate of change with respect to x.

0.2 Functions of 2 or more variables

Functions which have more than one variable arise very commonly. Simple examples are

- formula for the area of a triangle $A = \frac{1}{2}bh$ is a function of the two variables, base b and height h
- formula for electrical resistors in parallel:

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

  is a function of three variables R_1, R_2 and R_3, the resistances of the individual resistors.

Let's talk about functions of two variables here. You should be used to the notation $y = f(x)$ for a function of one variable, and that the graph of $y = f(x)$ is a curve. For functions of two variables the notation simply becomes

$$z = f(x, y)$$

where the two **independent** variables are x and y, while z is the **dependent** variable. The graph of something like $z = f(x, y)$ is a **surface** in three-dimensional space. Such graphs are usually quite difficult to draw by hand.
Since $z = f(x, y)$ is a function of two variables, if we want to differentiate we have to decide whether we are differentiating with respect to x or with respect to y (the answers are different). A special notation is used. We use the symbol ∂ instead of d and introduce the **partial derivatives** of z, which are:

- $\dfrac{\partial z}{\partial x}$ is read as "partial derivative of z (or f) with respect to x", and means differentiate with respect to x holding y constant
- $\dfrac{\partial z}{\partial y}$ means differentiate with respect to y holding x constant

Another common notation is the subscript notation:

$$z_x \quad \text{means} \quad \frac{\partial z}{\partial x}$$

$$z_y \quad \text{means} \quad \frac{\partial z}{\partial y}$$

Note that we cannot use the dash $'$ symbol for partial differentiation because it would not be clear what we are differentiating with respect to.

0.3 Example

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^2 + 3xy + y - 1$.

Solution. To find $\frac{\partial z}{\partial x}$ treat y as a constant and differentiate with respect to x. We have $z = x^2 + 3xy + y - 1$ so

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

0.4 Example

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = 1 - x - \frac{1}{2}y$. Interpret your answers and draw the graph.

Solution. The graph of $z = 1 - x - \frac{1}{2}y$ is a plane passing through the points $(x, y, z) = (1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$. The partial derivatives are:

$$\frac{\partial z}{\partial x} = -1, \qquad \frac{\partial z}{\partial y} = -\tfrac{1}{2}$$

Interpretation: $\frac{\partial z}{\partial x}$ is the slope you will notice if you walk on the surface in a direction keeping your y coordinate fixed. $\frac{\partial z}{\partial y}$ is the slope you will notice if you walk on the surface in such a direction that your x coordinate remains the same. There are, of course, many other directions you could walk, and the slope you will notice when walking in some other direction can be worked out knowing both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. It's like when you walk on a mountain, there are many directions you could walk and each one will have its own slope.

0.5 Other examples of evaluating partial derivatives

(i) $z = \ln(x^2 - y)$. Then $\frac{\partial z}{\partial x} = \frac{2x}{x^2 - y}$ and $\frac{\partial z}{\partial y} = \frac{-1}{x^2 - y}$. [To deduce these results we used the fact that if $y = \ln f(x)$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$].

(ii) $z = x\cos y + ye^x$. Then $\frac{\partial z}{\partial x} = \cos y + ye^x$ and $\frac{\partial z}{\partial y} = -x\sin y + e^x$.

(iii) $z = y\sin xy$. Then $\frac{\partial z}{\partial x} = y(y\cos xy) = y^2\cos xy$ and $\frac{\partial z}{\partial y} = yx\cos xy + \sin xy$. For the second result we used the product rule.

(iv) If $x^2 + y^2 + z^2 = 1$ find the rate at which z is changing with respect to y at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$. *Solution.* We have $z = (1 - x^2 - y^2)^{1/2}$. We want $\frac{\partial z}{\partial y}$ when

$(x, y) = (\frac{2}{3}, \frac{1}{3})$. But

$$\frac{\partial z}{\partial y} = \tfrac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y) = -\frac{y}{(1 - x^2 - y^2)^{1/2}}$$

Putting in $(x, y) = (\frac{2}{3}, \frac{1}{3})$ gives

$$\frac{\partial z}{\partial y} = -\frac{1/3}{(1 - (2/3)^2 - (1/3)^2)^{1/2}} = -\tfrac{1}{2}.$$

0.6 Functions of 3 or more variables

The general notation would be something like

$$w = f(x, y, z)$$

where x, y and z are the independent variables. For example, $w = x \sin(y + 3z)$. Partial derivatives are computed similarly to the two variable case. For example, $\partial w/\partial x$ means differentiate with respect to x holding both y and z constant and so, for this example, $\partial w/\partial x = \sin(y + 3z)$. Note that a function of three variables does not have a graph.

0.7 Second order partial derivatives

Again, let $z = f(x, y)$ be a function of x and y.

- $\dfrac{\partial^2 z}{\partial x^2}$ means the second derivative with respect to x holding y constant
- $\dfrac{\partial^2 z}{\partial y^2}$ means the second derivative with respect to y holding x constant
- $\dfrac{\partial^2 z}{\partial x \partial y}$ means differentiate first with respect to y and then with respect to x.

The "mixed" partial derivative $\dfrac{\partial^2 z}{\partial x \partial y}$ is as important in applications as the others. It is a general result that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

i.e. you get the same answer whichever order the differentiation is done.

0.8 Example

Let $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the first and second order partial derivatives of z.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 8x - 8y^4 \\
\frac{\partial z}{\partial y} &= -8x(4y^3) + 35y^4 = -32xy^3 + 35y^4 \\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 8 \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial y}(-32xy^3 + 35y^4) = -32x(3y^2) + 140y^3 \\
&= -96xy^2 + 140y^3 \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-32xy^3 + 35y^4) = -32y^3 \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x - 8y^4) = -32y^3
\end{aligned}$$

0.9 Example

Find all the first and second order partial derivatives of the function $z = \sin xy$.
Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y \cos xy \\
\frac{\partial z}{\partial y} &= x \cos xy \\
\frac{\partial^2 z}{\partial x^2} &= -y^2 \sin xy \\
\frac{\partial^2 z}{\partial y^2} &= -x^2 \sin xy \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x \cos xy) = x(-y \sin xy) + \cos xy = -xy \sin xy + \cos xy \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(y \cos xy) = y(-x \sin xy) + \cos xy = -xy \sin xy + \cos xy
\end{aligned}$$

0.10 Subscript notation for second order partial derivatives

If $z = f(x, y)$ then

- z_{xx} means $\frac{\partial^2 z}{\partial x^2}$
- z_{yy} means $\frac{\partial^2 z}{\partial y^2}$

- z_{xy} means $\dfrac{\partial^2 z}{\partial x \partial y}$ or $\dfrac{\partial^2 z}{\partial y \partial x}$

0.11 Important point

Unlike ordinary derivatives, partial derivatives do **not** behave like fractions, in particular

$$\frac{\partial x}{\partial z} \neq \frac{1}{\partial z / \partial x}$$

0.12 Small changes

Let

$$z = f(x, y)$$

Imagine we change x to $x + \delta x$ and y to $y + \delta y$ with δx and δy very small. We ask: what is the corresponding change in z? The answer is that the change is δz, given by

$$\delta z \approx \frac{\partial z}{\partial x}\, \delta x + \frac{\partial z}{\partial y}\, \delta y \tag{0.1}$$

This formula requires δx and δy to be very small and even then the formula is only an approximate one. However, it becomes more and more exact as $\delta x \to 0$ and $\delta y \to 0$. This fact is sometimes expressed by saying

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

where dx, dy and dz are infinitesimal increments.

Let's give some idea where formula (0.1) comes from. Let's recall the analogous result for a function of one variable and its derivation. For a function of one variable the notation would be $y = g(x)$ and the graph of this is a curve with a gradient dy/dx at each point x. If consider two points on this curve, (x, y) and a neighbouring point $(x + \delta x, y + \delta y)$ then if this neighbouring point is sufficiently close the line joining the two points, which has gradient $\delta y / \delta x$, is a good approximation to the tangent line at (x, y) which has gradient dy/dx. This means that $\delta y / \delta x \approx dy/dx$ so that $\delta y \approx (dy/dx)\delta x$.

We want to generalise this idea to a function $z = f(x, y)$ of two variables, whose graph will be a surface.

In the (x, y) plane let A be the point with coordinates (x, y), let B be the point with coordinates $(x + \delta x, y)$, and C the point with coordinates $(x + \delta x, y + \delta y)$.

The overall change in height, δz, from A to C is given by

$$\delta z = (\text{change in height } A \text{ to } B) + (\text{change in height } B \text{ to } C)$$

In calculating the change in height from A to B we are travelling across the surface from A to B along a curve in which y is held fixed, so by the result for curves,

$$\text{change in height } A \text{ to } B \approx \frac{\partial z}{\partial x}\, \delta x$$

Similarly

$$\text{change in height } B \text{ to } C \approx \frac{\partial z}{\partial y}\,\delta y$$

Therefore

$$\delta z \approx \frac{\partial z}{\partial x}\,\delta x + \frac{\partial z}{\partial y}\,\delta y$$

and we have derived formula (0.1).

0.13 Example

A cylindrical tank is 1 m high and 0.3 m radius. If height is increased by 5 cm and radius by 1 cm what is the effect on volume?

Solution. Let the radius be r and height be h. Then the volume V is given by

$$V = \pi r^2 h$$

so that $\frac{\partial V}{\partial r} = 2\pi r h$ and $\frac{\partial V}{\partial h} = \pi r^2$. Therefore in the notation of the present problem formula (0.1) becomes

$$\begin{aligned} \delta V &\approx \frac{\partial V}{\partial r}\,\delta r + \frac{\partial V}{\partial h}\,\delta h \\ &= 2\pi r h\,\delta r + \pi r^2 h\,\delta h \end{aligned}$$

In our case $r = 0.3$, $h = 1$, $\delta r = 1$ cm $= 0.01$ m, $\delta h = 5$ cm $= 0.05$ m so

$$\delta V \approx 2\pi(0.3)(1)(0.01) + \pi(0.3)^2(0.05) = 0.033 \ \ \mathrm{m}^3$$

0.14 Example

The angle of elevation of the top of a tower is found to be $30^o \pm 0.5^o$ from a point 300 ± 0.1 m from the base. Estimate the towers height.

Solution. One could imagine that this sort of problem would arise when a surveyor is unable to take completely accurate readings and wants to know the likely margin of error.

Let θ be the angle of elevation, h the towers height and x the distance from tower to observer. Then

$$h = x \tan\theta$$

so that $\frac{\partial h}{\partial x} = \tan\theta$ and $\frac{\partial h}{\partial \theta} = x\sec^2\theta$. Therefore

$$\begin{aligned} \delta h &\approx \frac{\partial h}{\partial x}\,\delta x + \frac{\partial h}{\partial \theta}\,\delta\theta \\ &= \tan\theta\,\delta x + x\sec^2\theta\,\delta\theta \end{aligned}$$

Now $\theta = 30^o = \pi/6$ radians and $\delta\theta = 0.5^o = 0.008727$ radians. Also $x = 300$ m and $\delta x = 0.1$ m. Therefore

$$\delta h \approx (\tan\pi/6)(0.1) + 300(\sec^2\pi/6)(0.008727) = 3.55 \text{ m}$$