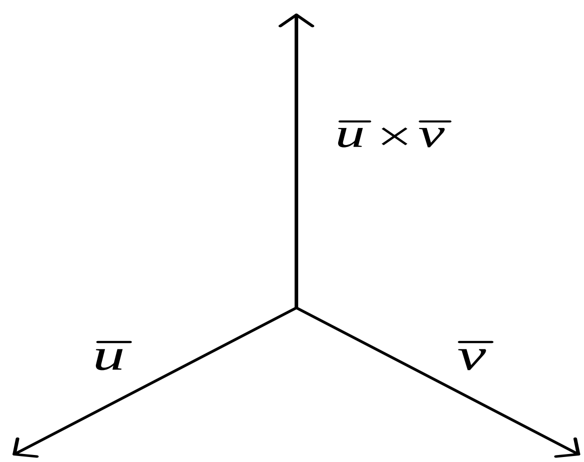
The Cross Products

- The cross product (vector product) $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to **u** and **v.** (illustrated in figure below)
- The direction is determined by the *right hand rule.*



✓ If the first two fingers of the right hand point in the directions of $\overline{u}$ and $\overline{v}$ respectively, then the thumb points in the direction of $\overline{u} \times \overline{v}$.

Ex: $\overline{i} \times \overline{j} = \overline{k}$

- The length is determined by the lengths of **u** and **v** and the angle between them.
- If we change the order informing the cross product, then we change the direction.

Ex:

$$\bar{v} \times \bar{u} = -(\bar{u} \times \bar{v})$$

-Theorem-

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,

then,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Properties of Cross Product

(a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(d) $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$

(e) **u** // **v** if and only if $\mathrm{u} \times \mathbf{v} = 0$

(f) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

Example:

1) Given that $\mathbf{u} = \langle 3,0,4 \rangle$ and $\mathbf{v} = \langle 1,5,-2 \rangle$, find

   (a) $\mathbf{u} \times \mathbf{v}$

   (b) $\mathbf{v} \times \mathbf{u}$

2) Find two unit vectors that are perpendicular to the vectors **u** =2**i**+2**j**-3**k** and **v** = **i**+3**j**+**k.**

Answer:

1) (a) -20**i**+10**j**+15**k** (b) 20**i**-10**j**-15**k**

2) $\pm\frac{1}{\sqrt{162}}\langle 11,-5,4\rangle$ (The unit vector in the opposite direction is also a unit vector perpendicular to both $\bar{u}$ and $\bar{v}$)

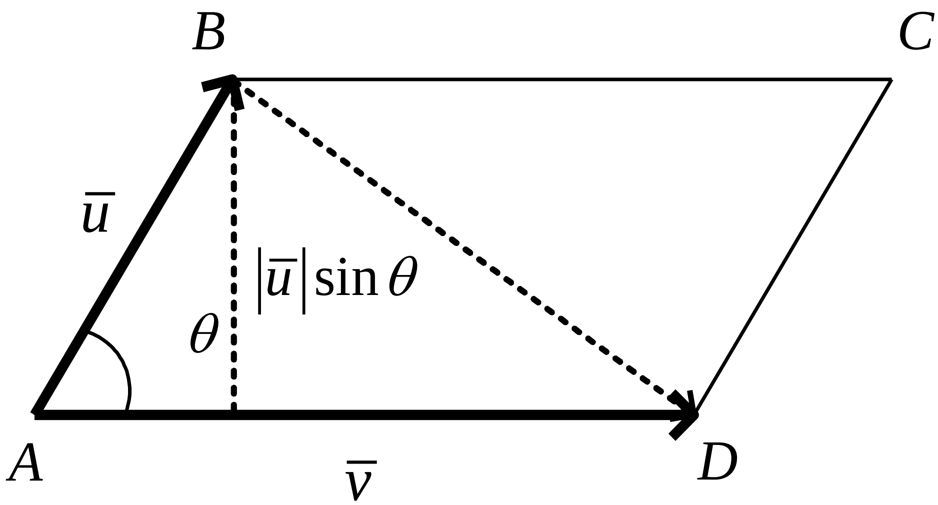
Further geometry interpretation of the cross product comes from computing its magnitude.

$$|\bar{u}\times\bar{v}|^2 = (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2$$

$$\begin{aligned}|\bar{u}\times\bar{v}|^2 &= \left(u_1{}^2+u_2{}^2+u_3{}^2\right)\left(v_1{}^2+v_2{}^2+v_3{}^2\right) \\ &\quad -(u_1v_1+u_2v_2+u_3v_3)^2 \\ &= |\bar{u}|^2|\bar{v}|^2-(\bar{u}\cdot\bar{v})^2 \\ &= |\bar{u}|^2|\bar{v}|^2-|\bar{u}|^2|\bar{v}|^2\cos^2\theta \\ &= |\bar{u}|^2|\bar{v}|^2\left(1-\cos^2\theta\right) \\ &= |\bar{u}|^2|\bar{v}|^2\sin^2\theta\end{aligned}$$

with θ is the angle between $\bar{u}$ and $\bar{v}$.

Therefore, $|\bar{u}\times\bar{v}|=|\bar{u}||\bar{v}|\sin\theta.$



From the figure above, we can see that the magnitude of the cross product is the area of the parallelogram of which arrows representing the two vectors are adjacent sides.

$$\text{Area of a parallelogram} = |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}\times\mathbf{v}|$$

$$\text{Area of triangle} = \frac{1}{2}|\mathbf{u}\times\mathbf{v}|$$

Example:

(a) Find an area of a parallelogram that is formed from vectors **u** = **i** + **j** - 3**k** and **v** = -6**j** + 5**k.**

(b) Find an area of a triangle that is formed from vectors **u** = **i** + **j** - 3**k** and **v** = -6**j** + 5**k.**

Answer:

(a) $\sqrt{230}$ (b) $\dfrac{\sqrt{230}}{2}$

Scalar Triple Product

-Theorem-

If $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$ and $\mathbf{c} = \langle x_3, y_3, z_3 \rangle$,

then

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Properties of The Scalar Triple Product

1) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

2) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

3) $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) = -\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

4) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$

5) $(\mathbf{a}+\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})$

Example:

If **a** = 3**i** + 4**j - k, b** = **-**6**j**+5**k** and **c** = **i**+**j-k,**

evaluate

(a) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ (b) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

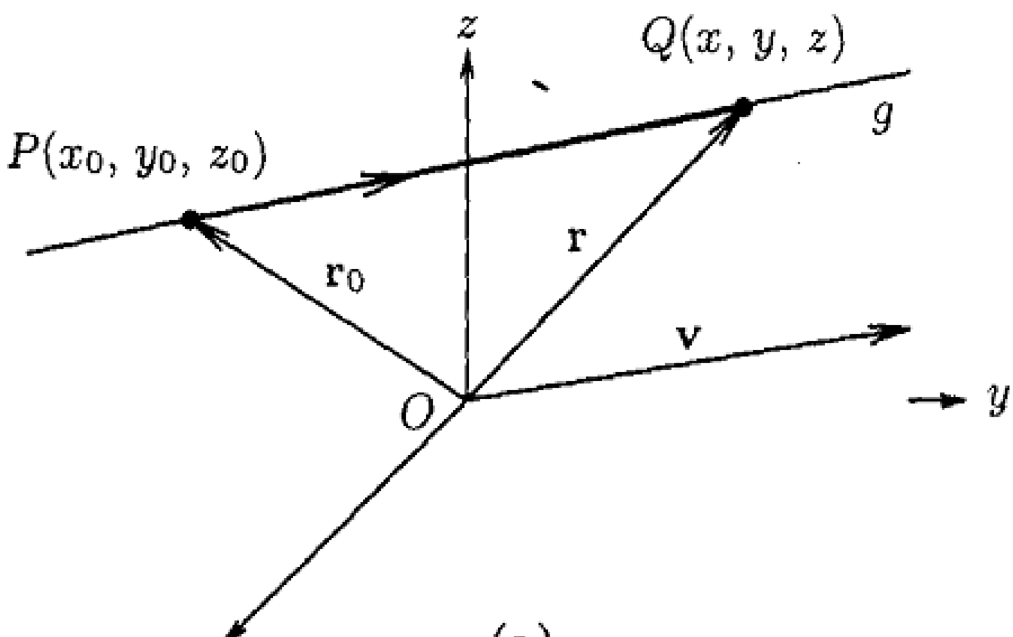
(c) $(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$ (d) $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$

6.1 Lines in Space

In this section we use vectors to study lines in three-dimensional space.

HOW LINES CAN BE DEFINED USING VECTORS?

The most convenient way to describe a line in space is to give a point on it and a nonzero vector parallel to it.



(a)

Suppose L is a straight line that passes through $P(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Thus, a point $Q(x, y, z)$ also lies on the line if

$$\overrightarrow{PQ} = t\text{v} \, .$$

Let,

$$\text{r}_0 = \overrightarrow{OP} \quad \text{and} \quad \text{r} = \overrightarrow{OQ} \, ,$$

Then

$$\therefore \overrightarrow{PQ} = \text{r} - \text{r}_0 \, .$$

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$$

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

$$< x, y, z > = < x_0, y_0, z_0 > + t < a, b, c >$$

-Theorem-

(Parametric Equations for a Line)

The line through the point $P(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{A} = \langle a, b, c \rangle$ has the **parametric equations,**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

If we let $\mathbf{R_0} = \langle x_0, y_0, z_0 \rangle$ denote the position vector of $P(x_0, y_0, z_0)$ and $\mathbf{R} = \langle x, y, z \rangle$ the position vector of the arbitrary point $Q(x,y,z)$ on the line, then we write equation (1) in the vector form,

$$\mathbf{R} = \mathbf{R}_0 + t\mathbf{A}.$$

Example:

Give the parametric equations for the line through the point (6,4,3) and parallel to the vector $\langle 2,0,-7 \rangle$.

-Theorem-

(Symmetric Equations for a line)

The line through the point $P(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{A} = \langle a, b, c \rangle$ has the **symmetrical equations,**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example:

Given that the symmetrical equations of a line in space is $\frac{2x+1}{3} = \frac{3-y}{4} = \frac{z+4}{2}$.

Find,

(a) a point on the line.

(b) a vector that is parallel to the line.