Note:

1. If $\sum|(-1)^n a_n|$ is converge then $\sum(-1)^n a_n$ is converge

If $\sum(-1)^n a_n$ is diverge then $\sum|(-1)^n a_n|$ is also diverge

The Absolutely & Conditional Convergence:

1. If $\sum(-1)^n a_n$ is convergence .this series is called **Absolutely Convergent** if $\sum|(-1)^n a_n|$ is converge.
2. If $\sum(-1)^n a_n$ is convergence and $\sum|(-1)^n a_n|$ is divergence then $\sum(-1)^n a_n$ is called **Conditional Convergent**

1. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is conv. but $\sum\left|\frac{(-1)^n}{n}\right| = \sum_{n=0}^{\infty}\frac{1}{n}$ is diverge

$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is **Conditionally Convergent**

2 – $\sum_{n=0}^{\infty}(-1)^n\frac{n}{n+1}$ is divergence because $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$

Power Series :

This has the form $\sum_{n=1}^{\infty} a_n(x-h)^n \;= a_1(x-h) + a_2(x-h)^2 + \; a_3(x-h)^3 \dots\dots..$

To study these series we find the interval of x for absolute convergence by using the ratio test .

EX: Find the interval of absolute convergence of :

1. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

Using ratio test $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| < 1 \quad = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| < 1$$

=0<1 for every value of x

$\therefore$ interval of conv. is $\infty > x > -\infty$

$\sum_{n=0}^{\infty} 3^n \frac{(x+5)^n}{4^n}$

Using ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\lim_{n\to\infty} \left| 3^{n+1} \frac{(x+5)^{n+1}}{4^{n+1}} . \ . \frac{4^n}{3^n (x+5)^n} \right| < 1 \quad = \lim_{n\to\infty} \left|\frac{3}{4}(x+5)\right| < 1$

= $-1 < \frac{3}{4}(x+5) < 1$

$\frac{-4}{3} <$ x+5 $< \frac{4}{3}$

$-\frac{19}{3} < x < \frac{-11}{3}$ radius of conv. R= 4/3

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$