6.1.1 Angle between Two Lines

Consider two straight lines

$$l_1 : \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

and

$$l_2 : \frac{x-x_2}{d} = \frac{y-y_2}{e} = \frac{z-z_2}{f}$$

The line l_1 parallel to the vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and the line l_2 parallel to the vector $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$. Since the lines l_1 and l_2 are parallel to the vectors **u** and **v** respectively, then the angle, θ between the two lines is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example:

Find an acute angle between line

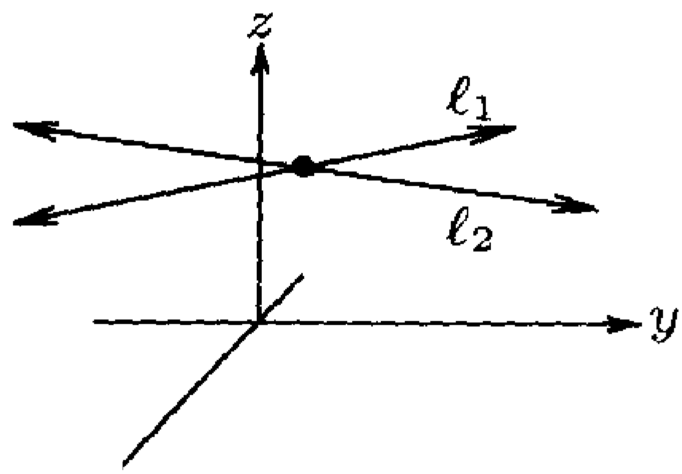
$$l_1 = \mathbf{i} + 2\mathbf{j} + t\,(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

and line

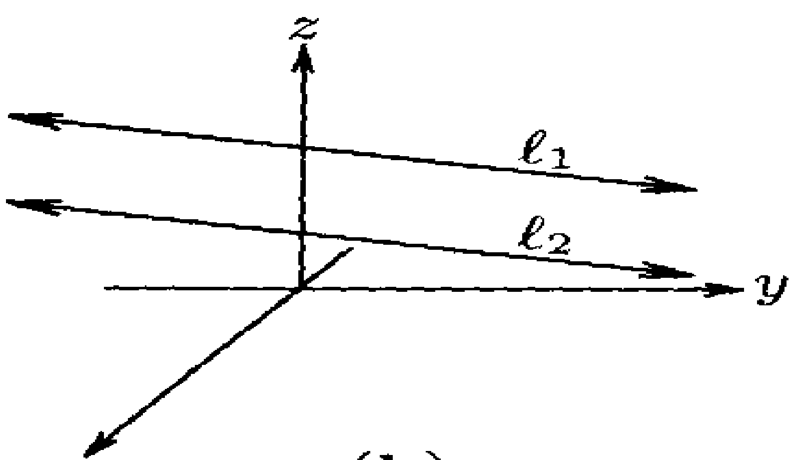
$$l_2 \quad = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + s\,(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}).$$

6.1.2 Intersection of Two lines

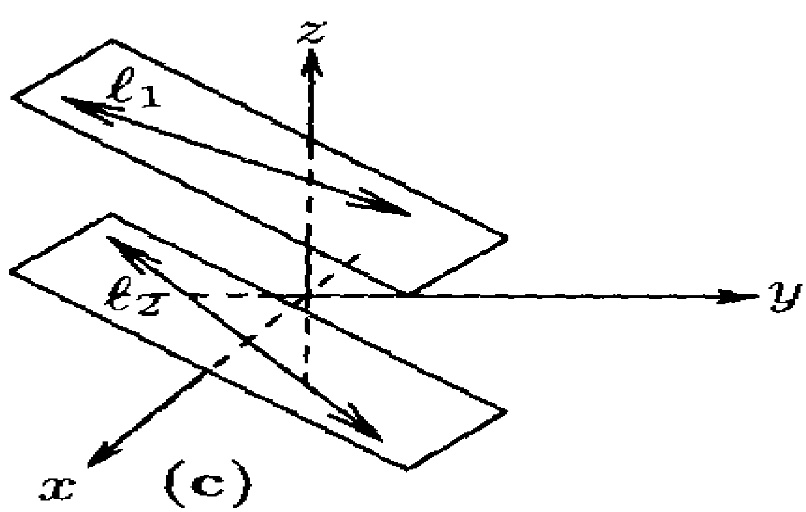
In three dimensional coordinates (space), two line can be in one of the three cases as shown below,



(a)



(b)



(c)

Let l_1 and l_2 are given by:

$$l_1: \ \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{and} \qquad (1)$$

$$l_2: \ \frac{x-x_2}{d} = \frac{y-y_2}{e} = \frac{z-z_2}{f} \qquad (2)$$

From (1), we have $\mathbf{v}_1 = < a, b, c >$

From (2), we have $\mathbf{v}_2 = < d, e, f >$

Two lines are <u>parallel</u> if we can write

$$\mathbf{v}_1 = \lambda \, \mathbf{v}_2$$

The parametric equations of l_1 and l_2 are:

$$l_1: \quad x = x_1 + a\,t$$
$$y = y_1 + b\,t$$
$$z = z_1 + c\,t$$

$$l_2: \quad \left.\begin{aligned} x &= x_2 + ds \\ y &= y_2 + es \\ z &= z_2 + fs \end{aligned}\right\} \quad (3)$$

Two lines are <u>intersect</u> if there exist unique values of t and s such that:

$$x_1 + at = x_2 + ds$$
$$y_1 + bt = y_2 + es$$
$$z_1 + ct = z_2 + fs$$

Substitute the value of t and s in (3) to get x, y and z. The <u>point of intersection</u> = (x, y, z)

Two lines are <u>skewed</u> if they are neither parallel nor intersect.

Example:

Determine whether l_1 and l_2 are parallel, intersect or skewed.

a) $l_1:\ x = 3 + 3t\ ,\ y = 1 - 4t,\ \ z = -4 - 7t$

$l_2:\ x = 2 + 3s\ ,\ y = 5 - 4s,\ \ z = 3 - 7s$

b) $l_1: \dfrac{x-1}{1} = \dfrac{2-y}{4} = z$

$l_2: \dfrac{x-4}{-1} = y-3 = \dfrac{z+2}{3}$

Solutions:

a) <u>for</u> l_1 :

point on the line, P = (3, 1, - 4)

vector that parallel to line, $\mathbf{v}_1 = <3, -4, -7>$

<u>for</u> l_2:

point on the line, Q = (2, 5, 3)

vector that parallel to line, $\mathbf{v}_2 = <3, -4, -7>$

$\mathbf{v}_1 = \lambda \mathbf{v}_2$?

$\mathbf{v}_1 = \mathbf{v}_2$ where $\lambda = 1$

Therefore, lines l_1 and l_2 are parallel.

b) Symmetrical equations of l_1 and l_2 can be rewrite as:

$$l_1: \quad \frac{x-1}{1} = \frac{y-2}{-4} = \frac{z-0}{1}$$

$$l_2: \quad \frac{x-4}{-1} = \frac{y-3}{1} = \frac{z-(-2)}{3}$$

Therefore:

<u>for</u> l_1 : P = (1, 2, 0) , $\mathbf{v}_1 = <1, -4, 1>$

<u>for</u> l_2: Q = (4, 3, -2) , $\mathbf{v}_2 = <-1, 1, 3>$

$\mathbf{v}_1 = \lambda \mathbf{v}_2$?

$\mathbf{v}_1 \neq \lambda \mathbf{v}_2 \rightarrow$ not parallel.

In parametric eq's:

$l_1: \ x = 1 + t \ , \ y = 2 - 4t \ , \ z = t$

$l_2: \ x = 4 - s \ , \ y = 3 + s \ , \ z = -2 + 3s$

$$1 + t = 4 - s \qquad (1)$$

$$2 - 4t = 3 + s \qquad (2)$$

$$t = -2 + 3s \qquad (3)$$

Solve the simultaneous equations (1), (2), and (3) to get t and s.

$$s = \frac{5}{4} \quad \text{and} \quad t = \frac{7}{4}$$

The value of t and s must satisfy (1), (2) and (3). Clearly they are not satisfying (2) i.e

$$2 - \frac{7}{4} = 3 + \frac{5}{4} \quad ? \quad \Rightarrow \quad \frac{1}{4} \neq \frac{17}{4}$$

Therefore, lines l_1 and l_2 are not intersect.

This implies the lines are skewed!

Example:

Let L_1 and L_2 be the lines

$$L_1 : x = 1 + 4t,\ y = 5 - 4t,\ z = -1 + 5t$$

$$L_2 : x = 2 + 8t,\ y = 4 - 3t,\ z = 5 + t$$

(a) Are the lines parallel?

(b) Do the lines intersect?