Sequences & Series

Sequences :

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

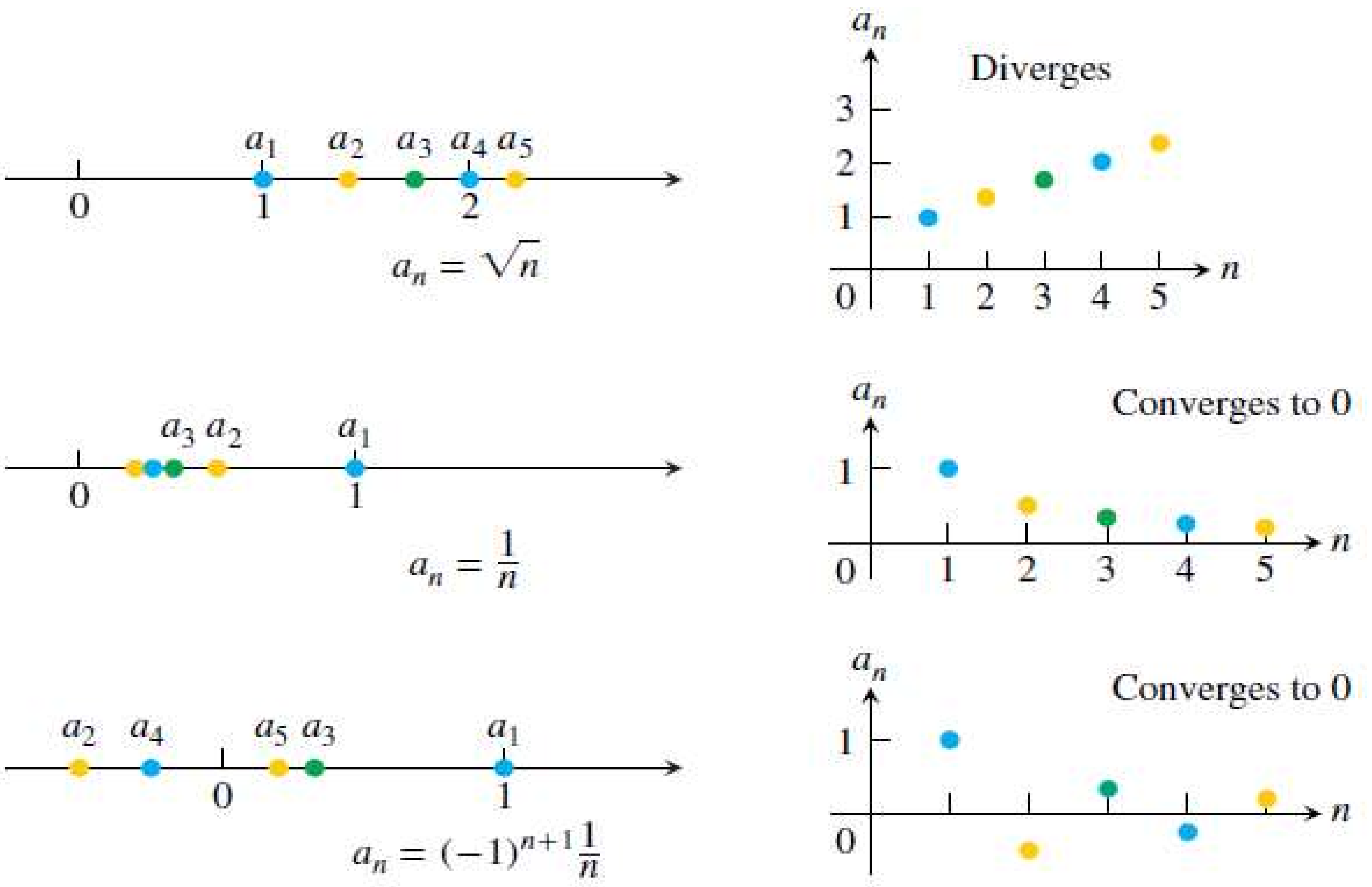
EX:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$



a1 a2 a3 a4 a5
0 1 2
an = √n
an
Diverges
3 2 1
0 1 2 3 4 5 n
a3 a2 a1
0 1
an = 1/n
an
Converges to 0
1
0 1 2 3 4 5 n
a2 a4 a5 a3 a1
0 1
an = (−1)^(n+1) 1/n
an
Converges to 0
1
0
n

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

EXAMPLE 1 Applying the Definition

Show that

(a) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim\limits_{n\to\infty} k = k$ (any constant k)

Solution

(a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty}(1/n) = 0$.

(b) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k. ■

EXAMPLE 2 A Divergent Sequence

Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

Solution Suppose the sequence converges to some number L. By choosing $\epsilon = 1/2$ in the definition of the limit, all terms a_n of the sequence with index n larger than some N must lie within $\epsilon = 1/2$ of L. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\epsilon = 1/2$ of L. It follows that $|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$. But the number L cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit L exists and so the sequence diverges.

Theorem:

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f\prime(x)}{g\prime(x)}$$ (L'Hopital's Rule)

يستخدم في حالة التعويض وينتج $\frac{0}{\infty}$,, $\frac{\infty}{\infty}$, $\frac{0}{0}$

EX: Use L'Hopital's Rule to find

$$\lim_{n\to\infty} \frac{2^n}{5n}.$$

By l'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$$
$$= \infty.$$

Applying L'Hôpital's Rule to Determine Convergence

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\begin{aligned}\ln a_n &= \ln\left(\frac{n+1}{n-1}\right)^n \\ &= n \ln\left(\frac{n+1}{n-1}\right).\end{aligned}$$

Then,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2.

The following six sequences converge to the limits listed below:

1. $\lim_{n \to \infty} \frac{\ln n}{n} = 0$

2. $\lim_{n \to \infty} \sqrt[n]{n} = 1$

3. $\lim_{n \to \infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n \to \infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Ex: Check the convergence of the following sequences :

1- $a_n = \sqrt{\frac{n+1}{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty}(1+\frac{1}{n})} = \sqrt{1+\frac{1}{\infty}} = \sqrt{1+0}$$

$$= 1\ (conv.)$$

2-$a_n = (1-\frac{3}{x})^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-\frac{3}{x})^n = \lim_{n\to\infty}(1+\frac{-3}{x})^n = e^{-3}$ conv. (from 5)