Vectors in Space

Properties of Vectors in Space

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be vectors in 3 dimensional space and k is a constant.

1. $\mathbf{v} = \mathbf{w}$ if and only if

$$v_1 = w_1, v_2 = w_2, v_3 = w_3.$$

2. The magnitude of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2}$

3. The unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle v_1, v_2, v_3 \rangle}{|\mathbf{v}|}$$

4. $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

5. $k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$

6. Zero vector is denoted as $\mathbf{0} = \langle 0,0,0 \rangle$.

7. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

8. Let $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$,

then $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

9. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

10. $\mathbf{u} + (\mathbf{-u}) \mathbf{= 0}$

11. $(\mathbf{c + d})\mathbf{u} \mathbf{= cu} + \mathrm{d}\mathbf{u}$

12. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

13. $\mathrm{c}(\mathrm{d}\mathbf{u}) = (\mathrm{cd})\mathbf{u}$

14. $1(\mathbf{u}) \mathbf{= u}$ and $0(\mathbf{u}) \mathbf{= 0}$

15. $|\mathrm{c}\mathbf{u}| = \mathrm{c}|\mathbf{u}|$

Example:

Express the vector $\overrightarrow{PQ}$ if it starts at point $P = \mathbf{(}6\mathbf{,}5\mathbf{,}8\mathbf{)}$ and stops at point $Q = \mathbf{(}7\mathbf{,}3\mathbf{,}9\mathbf{)}$ in components form.

Solution:

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$$\overrightarrow{PQ} = \langle 7-6, 3-5, 9-8 \rangle$$

$$\overrightarrow{PQ} = \langle 1, -2, 1 \rangle$$

Example:

Given that $\mathbf{a} = \langle 3,1,-2 \rangle$, $\mathbf{b} = \langle -1,6,4 \rangle$. Find

(a) $\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$

(c) a unit vector which is in the direction of **b**.

(d) find the unit vector which has the same direction as $\mathbf{a} + 3\mathbf{b}$.

Answer:

(a) $\langle 0,19,10 \rangle$ (b) $\sqrt{53}$ (c) $\left\langle -2, \frac{41}{6}, \frac{23}{3} \right\rangle$

(d) 1 (e) $\frac{1}{\sqrt{53}} \langle -1, 6, 4 \rangle$ (f) $\frac{1}{\sqrt{461}} \langle 0,19,10 \rangle$

<u>Parallel Vector</u>

- ✓ have same slopes
- ✓ $v_1 = \lambda v_2$; λ constants

So, there are multiples of each other.

Example:

Vector **w** has initial point (2,-1,3) and terminal point (-4,7,5). Which of the following vectors is parallel to **w**?

Solution:

$$\bar{w} = \langle -4-2, 7+1, 5-3 \rangle$$

$$\bar{w} = \langle -6, 8, 2 \rangle$$

One example: $2\langle -6, 8, 2 \rangle = \langle -12, 16, 4 \rangle$

Another is $\frac{1}{2}\langle -6, 8, 2 \rangle = \langle -3, 4, 1 \rangle$.

Example: (Collinear Points)

Determine whether the point $P(1,-2,3), Q(2,1,0)$ and $R(4,7,-6)$ lie on the same line.

Solution:

$$\overrightarrow{PQ} = \langle 2-1, 1+2, 0-3 \rangle = \langle 1, 3, -3 \rangle$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{1^2 + 3^2 + (-3)^2} = \sqrt{19}$$

$$\overrightarrow{QR} = \langle 4-2, 7-1, -6-0 \rangle = \langle 2, 6, -6 \rangle$$

$$\left|\overrightarrow{QR}\right| = \sqrt{2^2 + 6^2 + (-6)^2} = 2\sqrt{19}$$

$$\overrightarrow{PR} = \langle 4-1, 7+2, -6-3 \rangle = \langle 3, 9, -9 \rangle$$

$$\left|\overrightarrow{PR}\right| = \sqrt{3^2 + 9^2 + (-9)^2} = 3\sqrt{19}$$

Thus, $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$. Since one vector is a multiple of the other, the two vectors are parallel and since they share a common point Q, they must be the same line.

The Dot Product

-Theorem-

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then the scalar product $\mathbf{v} \cdot \mathbf{w}$ is

$$\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle$$

$$= v_1 w_1 + v_2 w_2 + v_3 w_3$$

-Note-

The dot product is also called

- the scalar product
- the inner product

The dot product of two vectors is a scalar.

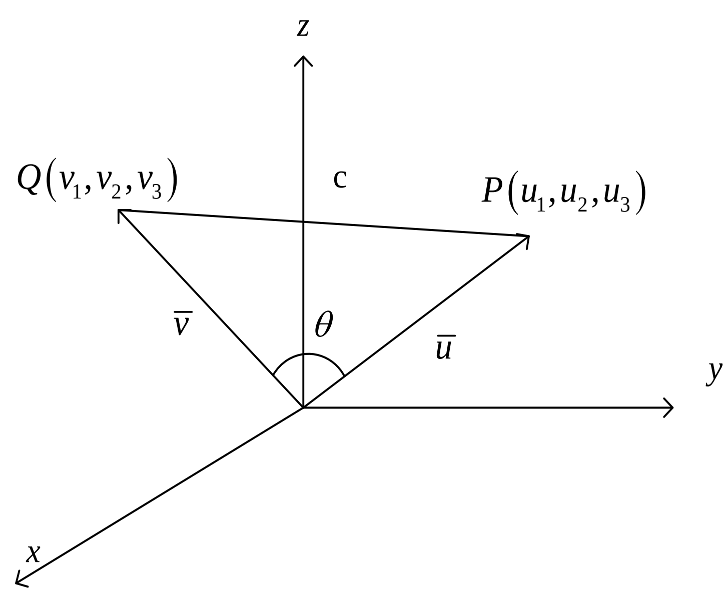
The Angle between Vectors

Refer to the figure below, let

$$\bar{u} = \bar{u}(OP) = \langle u_1, u_2, u_3 \rangle,$$

$$\bar{v} = \bar{v}(OQ) = \langle v_1, v_2, v_3 \rangle$$

be two vectors and let θ be the angle between them, with $0 \le \theta \le \pi$.



Compute the distance, *c* between points *P* and *Q* in two ways.

1) *Using the Distance formula*

$$c^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

$$= u_1{}^2 + u_2{}^2 + u_3{}^2 + v_1{}^2 + v_2{}^2 + v_3{}^2$$
$$- 2(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

$$= |\overline{u}|^2 + |\overline{v}|^2 - 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \quad \text{---(1)}$$

2) *Using the Law of Cosines*

$$c^2 = |\overline{u}|^2 + |\overline{v}|^2 - 2|\overline{u}||\overline{v}|\cos\theta \quad \text{---(2)}$$

Equating equation (1) and (2), we get

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\overline{u}||\overline{v}|} = \frac{\overline{u} \cdot \overline{v}}{|\overline{u}||\overline{v}|}$$

Example:

If **v** = 2**i**-**j**+**k**, **w** = **i**+**j**+2**k** and the angle between **v** and **w** is 60°, find $\mathbf{v} \cdot \mathbf{w}$.

Solution:

$$\bar{v}\cdot\bar{w}=\sqrt{2^2+(-1)^2+1^2}\cdot\sqrt{1^2+1^2+2^2}\cos(\pi/3)$$

$$=\sqrt{6}\cdot\sqrt{6}\cos(\pi/3)=6(1/2)=3$$

Example:

Given that $\mathbf{u}=\langle 2,-2,3\rangle$, $\mathbf{v}=\langle 5,8,1\rangle$ and $\mathbf{w}=\langle -4,3,-2\rangle$, find

(a) $\mathbf{u}\cdot\mathbf{v}$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

(c) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$

(d) the angle between **u** and **v**

(e) the angle between **v** and **w.**

Answer:

(a) -3

(b) $\langle 12,-9,6\rangle$

(c) -23

(d) $94^\circ 24'$

(e) $87^\circ 46'$

Example:

Let A=(4,1,2), B=(3,4,5) and C=(5,3,1) are the vertices of a triangle. Find the angle at vertex A.

Answer:

$79^\circ 12'$

-Theorem-

The nature of an angle θ, between two vectors **u** and **v**.

- ✓ θ is an acute angle if and only if $\mathbf{u} \cdot \mathbf{v} > 0$
- ✓ θ is an obtuse angle if and only if $\mathbf{u} \cdot \mathbf{v} < 0$
- ✓ $\theta = 90^\circ$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The Vectors **u** and **v** are orthogonal / perpendicular.

Example:

Show that the given vectors are perpendicular to each other.

(a) **i** and **j**

(b) 3**i**-7**j**+2**k** and 10**i**+4**j**-**k**

-Theorem-

(Properties of Dot Product)

If **u**,**v** and **w** are nonzero vectors and *k* is a scalar,

1. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

3. $k\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot k\mathbf{v}$

4. $\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2$

5. $\mathbf{u}\cdot\mathbf{0}=\mathbf{0}\cdot\mathbf{u}=0$