Ministry of Higher Education and Scientific Research
Almustaqbal University, College of Engineering
And Engineering Technologies
Computer Technology Engineering Department

Three week :

Circuit Elements in the Phasor Domain

Course Name : Fundamentals of Electricity
Stage : One
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Assist. Prof. Zahraa Hazim Al-Fatlawi

zahraa.hazim@uomus.edu.iq

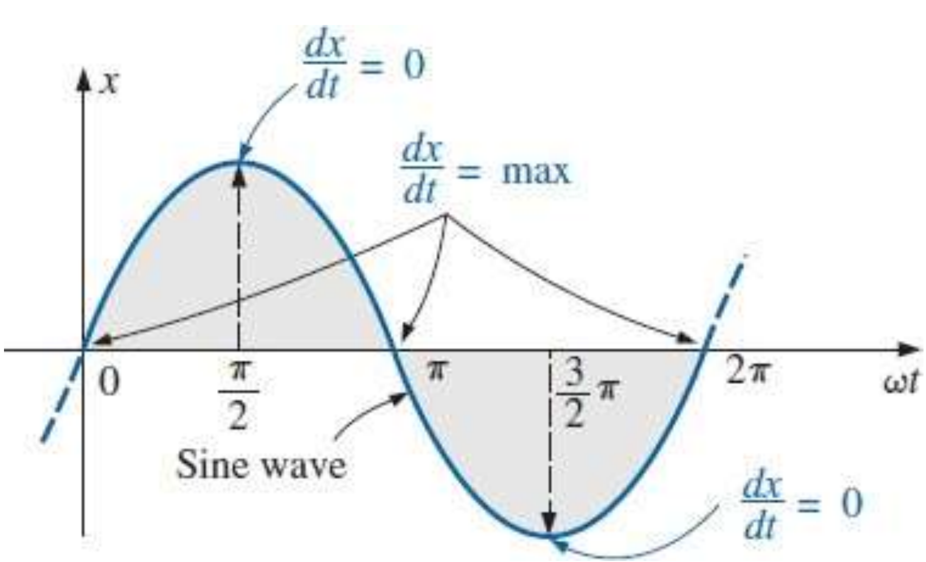
THE BASIC ELEMENTS AND PHASORS

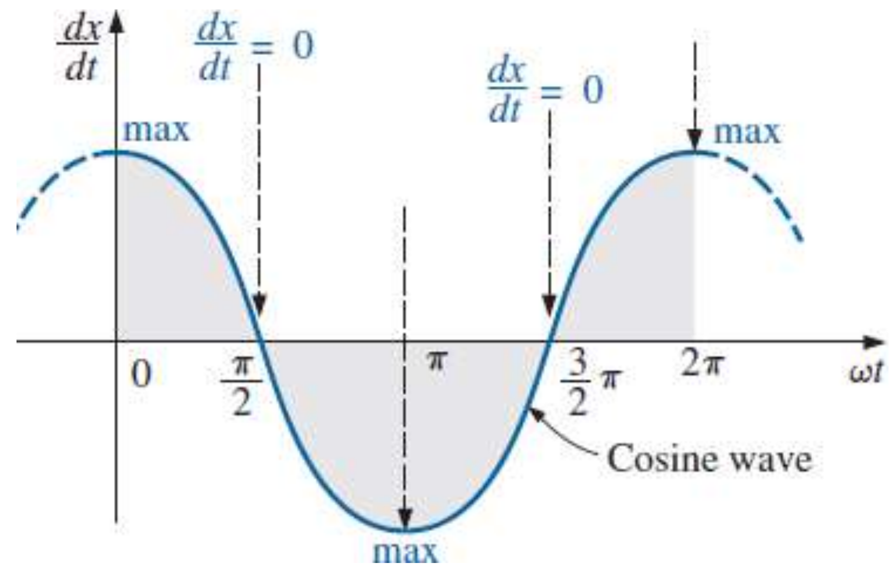
14.2 DERIVATIVE

To understand the response of the basic R, L, and C elements to a sinusoidal signal, you need to examine the concept of the **derivative** in some detail.

The derivative dx/dt is defined as the rate of change of x with respect to time. If x fails to change at a particular instant, dx = 0, and the derivative is zero.

For the sinusoidal waveform, dx/dt is zero only at the positive and negative peaks (ωt = π/2 and 3/2π in Fig.), since x fails to change at these instants of time.

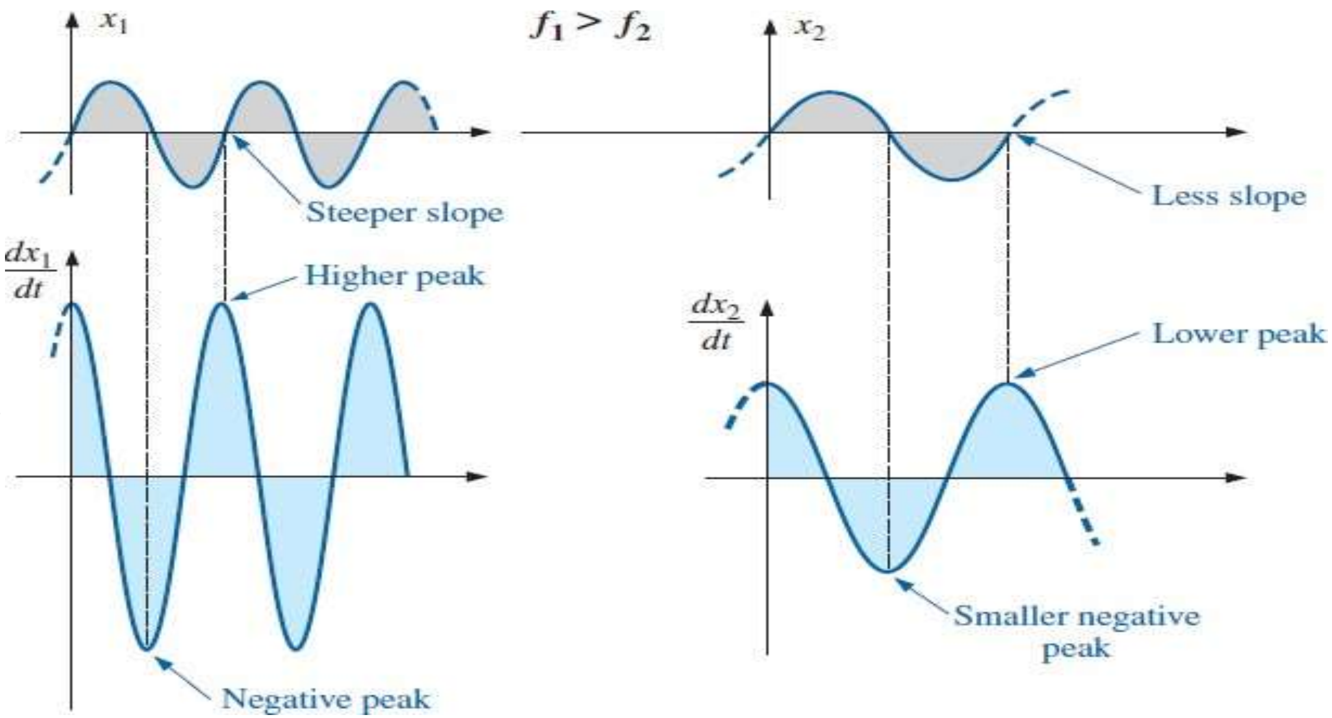




the derivative of a sine wave is a cosine wave.

Note in Fig.

that even both though waveforms (x1 and x2) have value the same peak the sinusoidal with the frequency the larger function higher produces peak value for the derivative



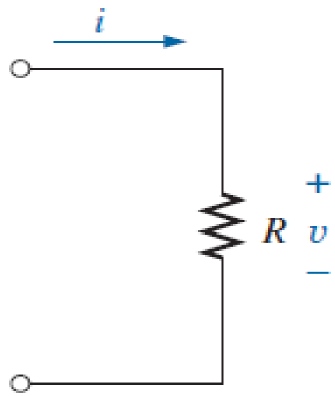
The derivative of a sine wave has the same period and frequency as the original sinusoidal waveform.

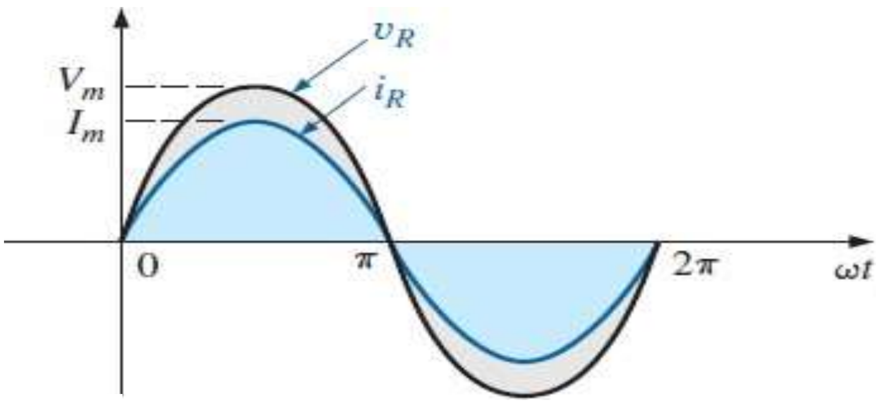
$$e(t) = E_m \sin(\omega t \pm \theta)$$

$$\frac{d}{dt}e(t) = \omega E_m \cos(\omega t \pm \theta) = 2\pi f E_m \cos(\omega t \pm \theta)$$

14.3 RESPONSE OF BASIC *R, L, AND C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

Resistor





$$i = \frac{v}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t$$

where $$I_m = \frac{V_m}{R}$$

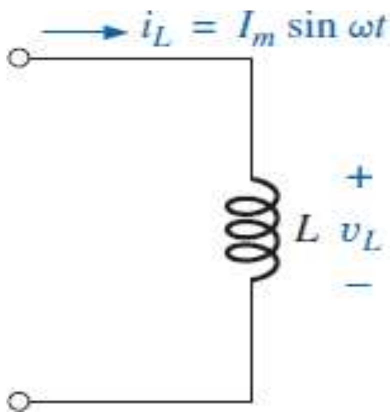
For a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm's law.

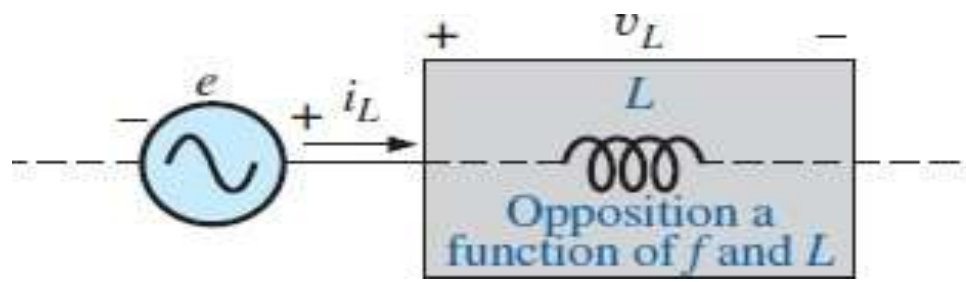
In addition, for a given i,

$$v = iR = (I_m \sin \omega t)R = I_m R \sin \omega t = V_m \sin \omega t$$

where $$V_m = I_m R$$

Inductor



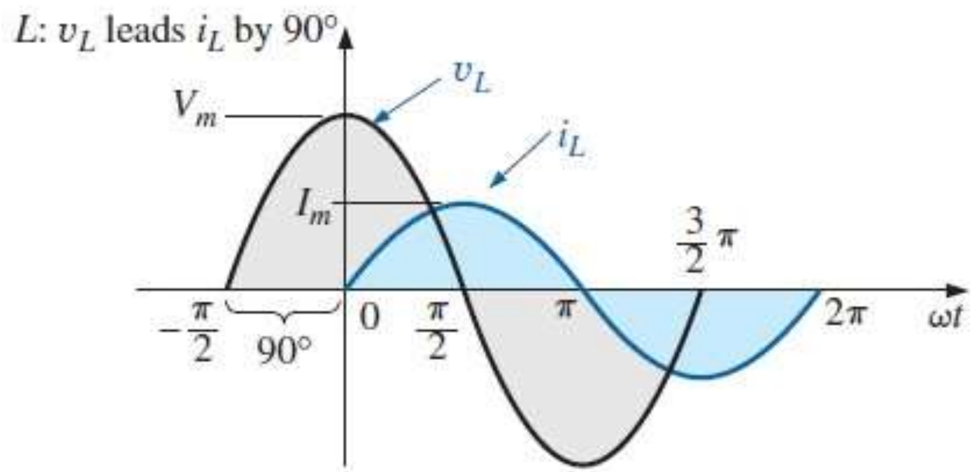


$$v_L = L\frac{di_L}{dt}$$

and, applying differentiation,

$$\frac{di_L}{dt} = \frac{d}{dt}(I_m \sin \omega t) = \omega I_m \cos \omega t$$

$$v_L = L\frac{di_L}{dt} = L(\omega I_m \cos \omega t) = \omega L I_m \cos \omega t$$



For an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°.

or $\quad v_L = V_m \sin(\omega t + 90°)$

where $\quad V_m = \omega L I_m$

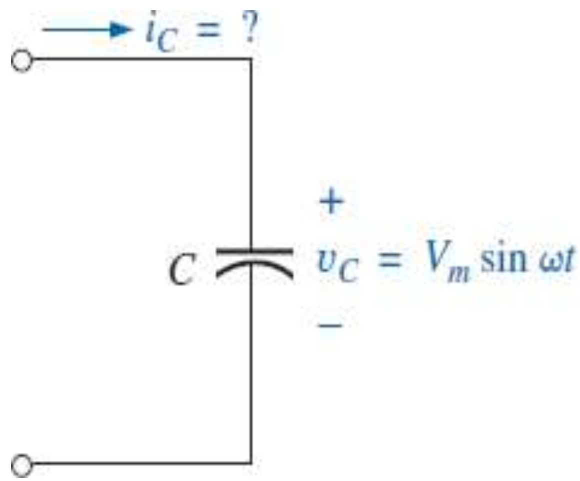
$$i_L = I_m \sin(\omega t \pm \theta)$$

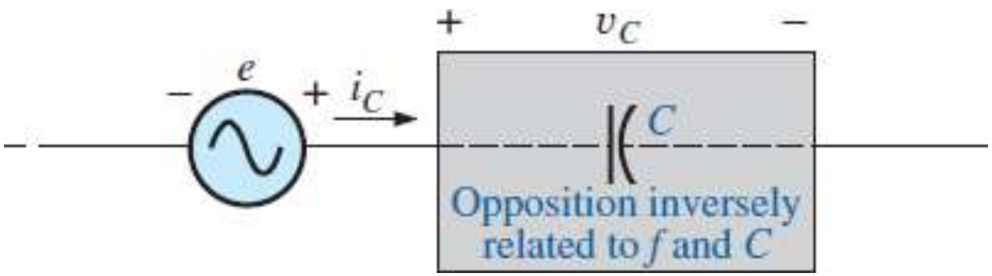
$$v_L = \omega L I_m \sin(\omega t \pm \theta + 90°)$$

The quantity **ωL**, called the **reactance** (from the word reaction) of an inductor, is symbolically represented by **X_L** and is measured in ohms; that is,

$$\boxed{X_L = \omega L} \quad \text{(ohms, } \Omega\text{)} \qquad \boxed{X_L = \frac{V_m}{I_m}} \quad \text{(ohms, } \Omega\text{)}$$

Capacitor

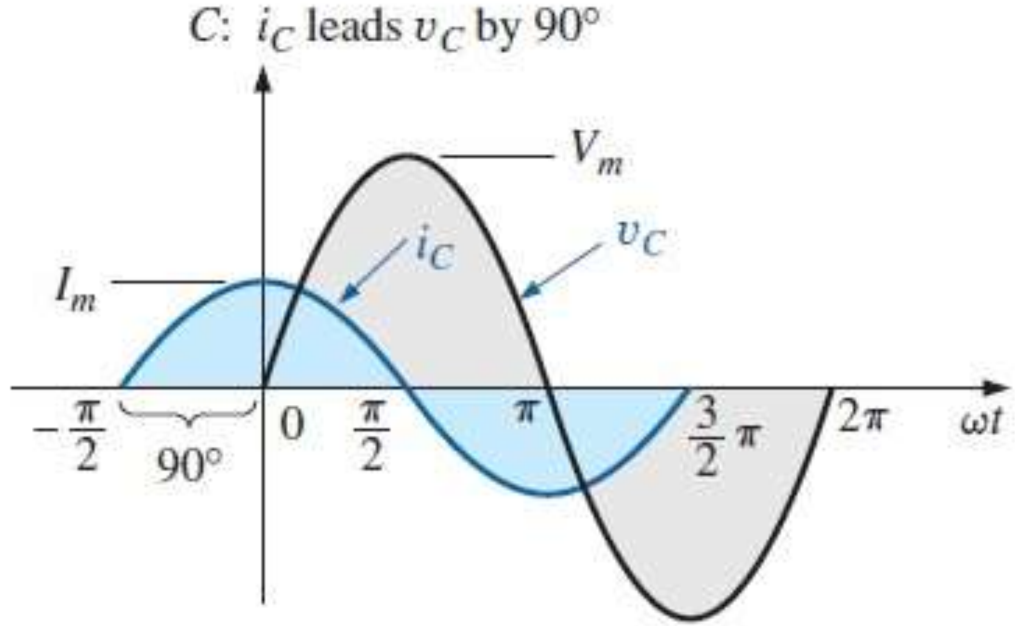




For a particular change in voltage across the capacitor, the greater the value of capacitance, the greater the resulting capacitive current.

In addition, the fundamental equation relating the voltage across a capacitor to the current of a capacitor $[i = C(dv/dt)]$ indicates that

Since an increase in current corresponds to a decrease in opposition, and ic is proportional to ω and C, the opposition of a capacitor is inversely related to ω (ω =2πf) and C.



$$i_C = C\frac{dv_C}{dt}$$

and, applying differentiation,

$$\frac{dv_C}{dt} = \frac{d}{dt}(V_m \sin \omega t) = \omega V_m \cos \omega t$$

For a capacitor, i_c leads v_c by 90°, or v_c lags i_c by 90°.*

Therefore,

$$i_C = C\frac{dv_C}{dt} = C(\omega V_m \cos \omega t) = \omega C V_m \cos \omega t$$

$$v_C = V_m \sin(\omega t \pm \theta)$$

$$i_C = \omega C V_m \sin(\omega t \pm \theta + 90°)$$

or $\quad i_C = I_m \sin(\omega t + 90°)$

where $\quad I_m = \omega C V_m$

The quantity $1/\omega C$, called the **reactance** of a capacitor, is symbolically represented by X_C and is measured in ohms; that is,

$$X_C = \frac{1}{\omega C} \quad \text{(ohms, } \Omega\text{)} \qquad X_C = \frac{V_m}{I_m} \quad \text{(ohms, } \Omega\text{)}$$

In the inductive circuit, $v_L = L\frac{di_L}{dt}$ In the capacitive circuit, $i_C = C\frac{dv_C}{dt}$

but $$i_L = \frac{1}{L}\int v_L\,dt$$ but $$v_C = \frac{1}{C}\int i_C\,dt$$

If the source current leads the applied voltage, the network is predominantly capacitive, and if the applied voltage leads the source current, it is predominantly inductive.

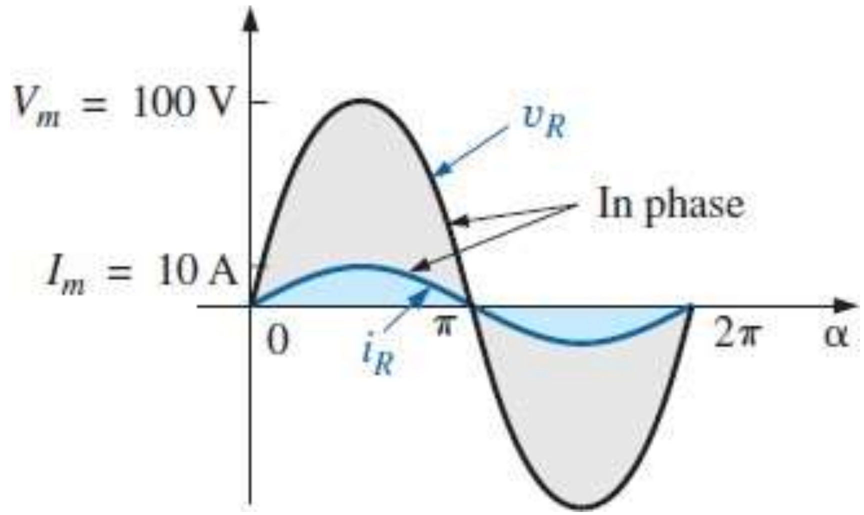
EXAMPLE 14.1 The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10Ω . Sketch the curves for v and *i.*

a. $v = 100 \sin 377t$
b. $v = 25 \sin(377t + 60°)$

a. $$I_m = \frac{V_m}{R} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$$

(v and i are in phase), resulting in

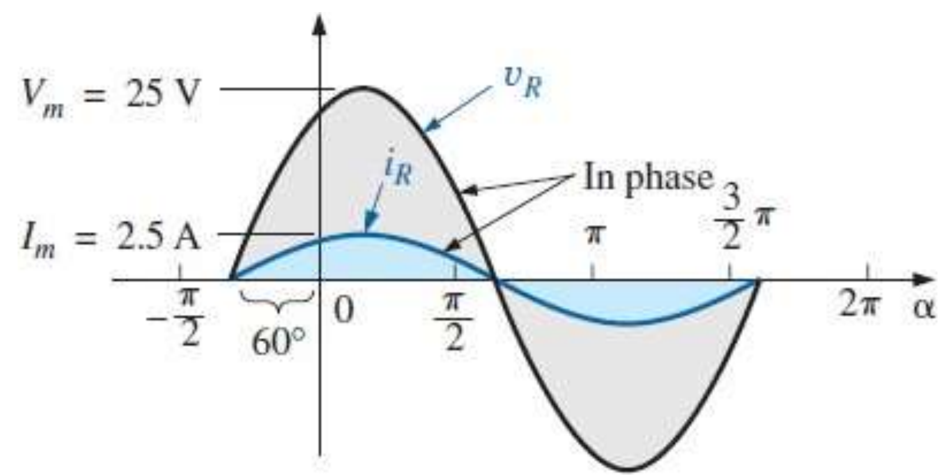
$$i = \mathbf{10 \sin 377t}$$



b. $$I_m = \frac{V_m}{R} = \frac{25\text{ V}}{10\ \Omega} = 2.5\text{ A}$$

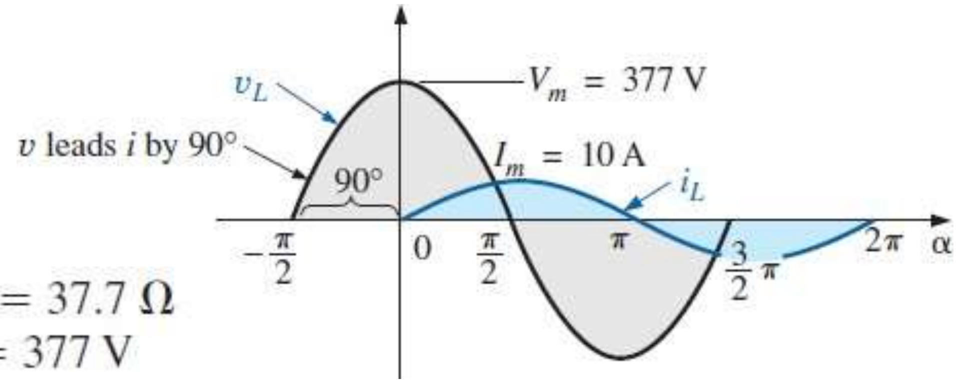
(v and i are in phase), resulting in

$$i = \mathbf{2.5 \sin(377t + 60°)}$$



EXAMPLE 14.3 The current through a 0.1 H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the *v* and *i curves.*

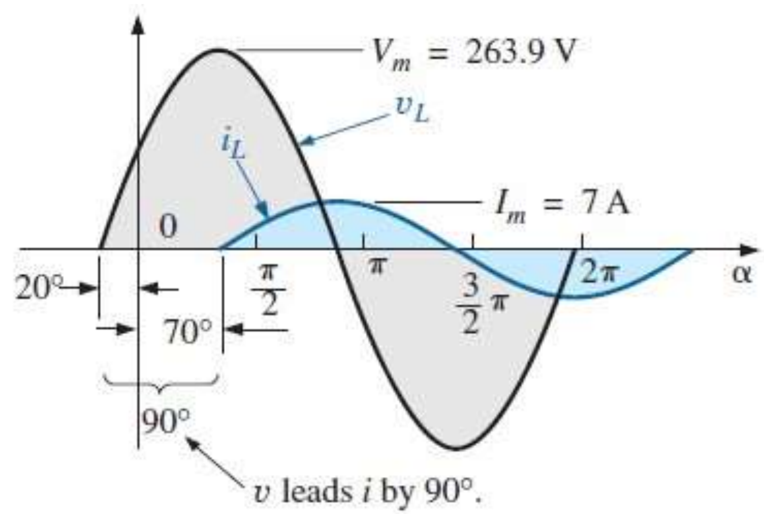
a. $i = 10 \sin 377t$
b. $i = 7 \sin(377t - 70°)$



a. $$X_L = \omega L = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7\ \Omega$$
$$V_m = I_m X_L = (10 \text{ A})(37.7\ \Omega) = 377 \text{ V}$$

and we know that for a coil v leads i by 90°. Therefore,

$$v = \mathbf{377 \sin(377t + 90°)}$$



b. X_L remains at 37.7 Ω.

$$V_m = I_m X_L = (7 \text{ A})(37.7\ \Omega) = 263.9 \text{ V}$$

and we know that for a coil v leads i by 90°. Therefor

$$v = 263.9 \sin(377t - 70° + 90°)$$

and

$$v = \mathbf{263.9 \sin(377t + 20°)}$$

EXAMPLE 14.6 The current through a 100 μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

$$i = 40 \sin(500t + 60°)$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^6 \ \Omega}{5 \times 10^4} = \frac{10^2 \ \Omega}{5} = 20 \ \Omega$$

$$V_M = I_M X_C = (40 \text{ A})(20 \ \Omega) = 800 \text{ V}$$

and we know that for a capacitor, v lags i by 90°. Therefore,

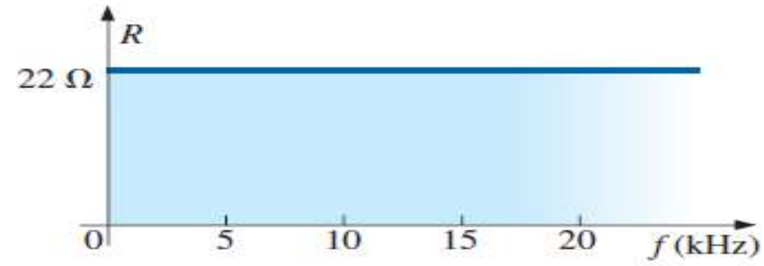
$$v = 800 \sin(500t + 60° - 90°)$$

and $$v = \mathbf{800 \sin(500t - 30°)}$$

14.4 FREQUENCY RESPONSE OF THE BASIC ELEMENTS

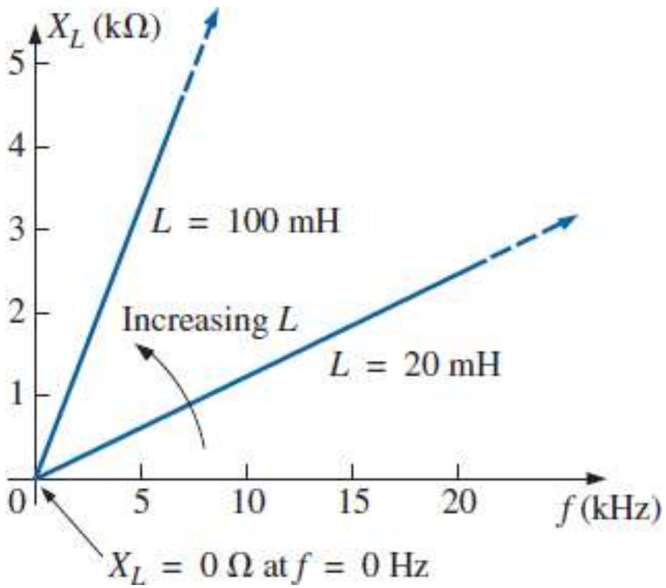
Ideal Response

Resistor *R*



R versus f for the range of interest.

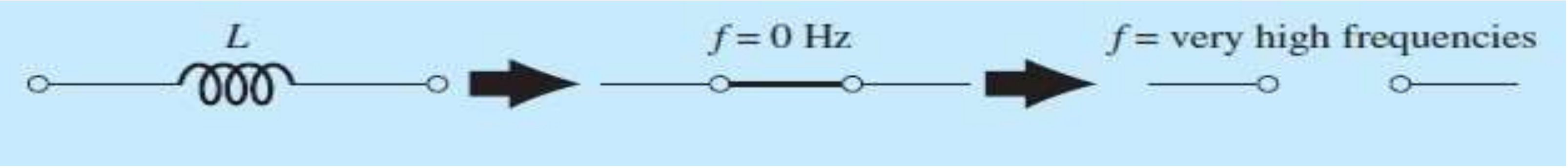
Inductor *L*



Straight Line Equation

$$X_L = \omega L = 2\pi f L = (2\pi L)f = kf \quad \text{with } k = 2\pi L$$

At a frequency of 0 Hz, an inductor takes on the characteristics of a short circuit, as shown in Fig., at very high frequencies, the characteristics of an inductor approach those of an open circuit.



Capacitor C

For the capacitor, the equation for the reactance

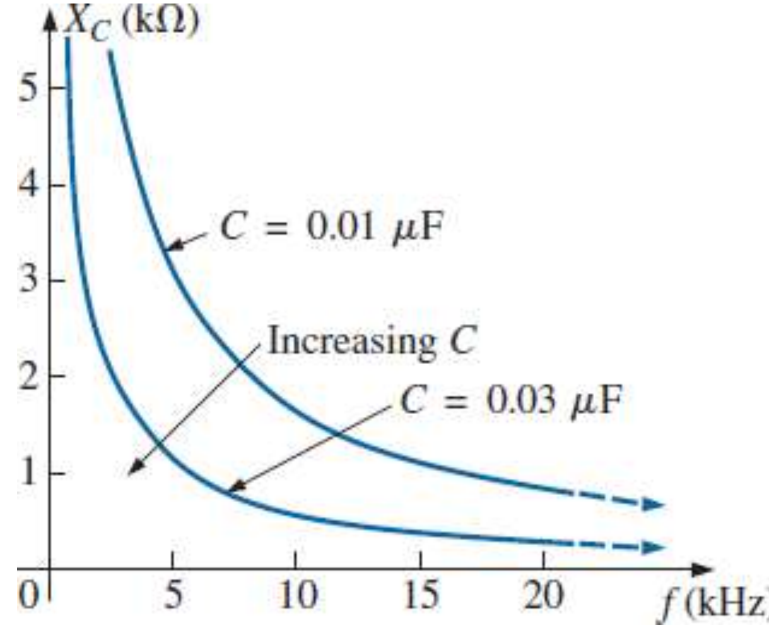
$$X_C = \frac{1}{2\pi f C}$$

can be written as

$$X_C f = \frac{1}{2\pi C} = k \qquad \text{(a constant)}$$

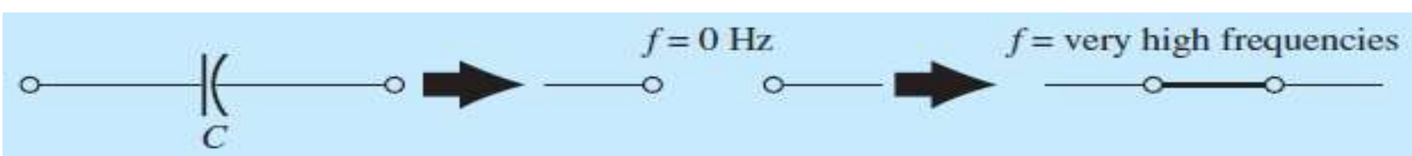
which matches the basic format for a hyberbola:

$$yx = k$$



$$X_C = \frac{1}{2\pi f c} = \frac{1}{2\pi(0\ \text{Hz})C} \Rightarrow \infty\ \Omega$$

At or near 0 Hz, the characteristics of a capacitor approach those of an open circuit, as shown in Fig., at very high frequencies, a capacitor takes on the characteristics of a short circuit



EXAMPLE 14.8 At what frequency will the reactance of a 200 mH inductor match the resistance level of a 5 kΩ resistor?

Solution: The resistance remains constant at 5 kΩ for the frequency range of the inductor. Therefore,

$$R = 5000\ \Omega = X_L = 2\pi f L = 2\pi L f$$
$$= 2\pi(200 \times 10^{-3}\ \text{H})f = 1.257f$$

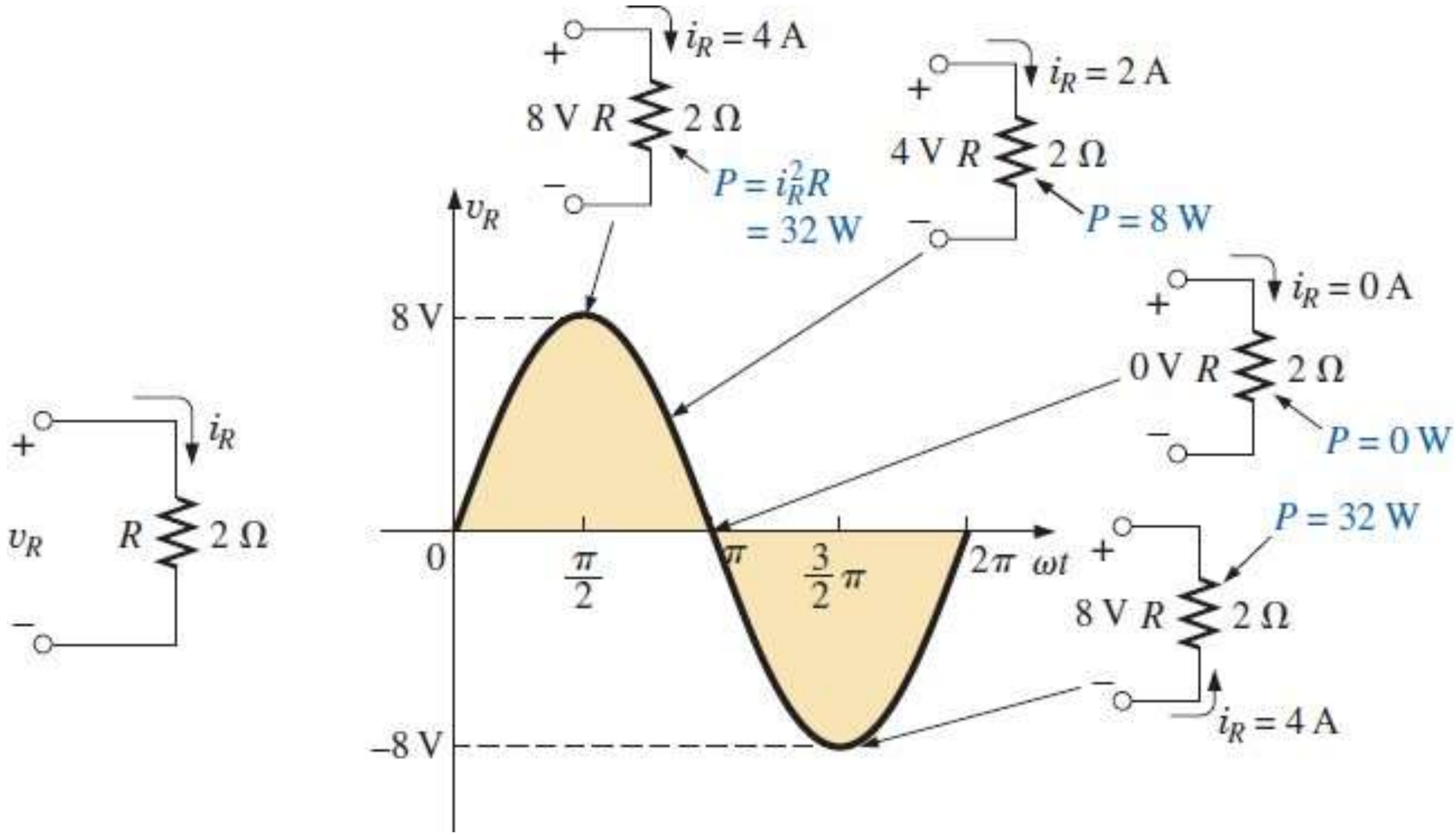
and
$$f = \frac{5000}{1.257} \cong \mathbf{3.98\ kHz}$$

EXAMPLE 14.9 At what frequency will an inductor of 5 mH have the same reactance as a capacitor of 0.1 *μF?*

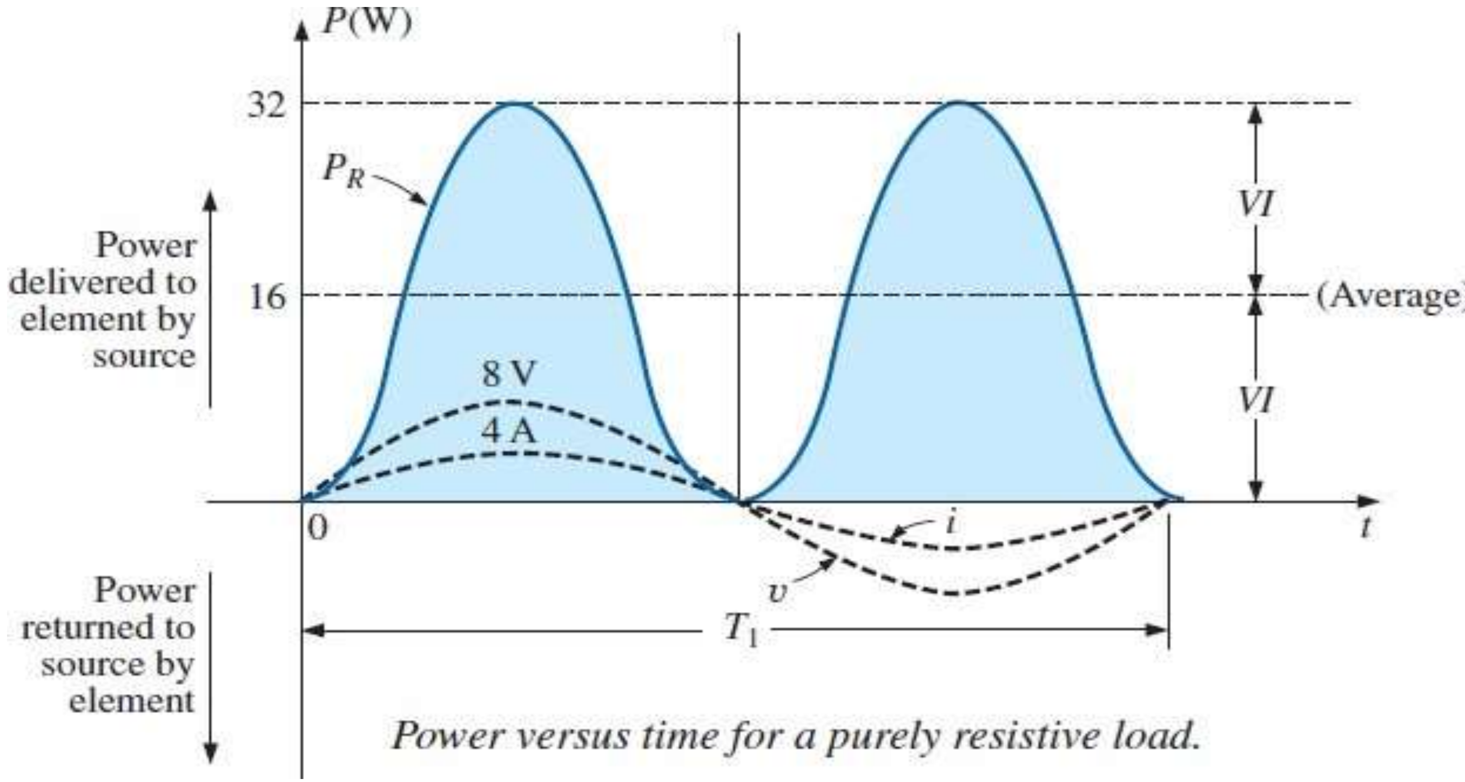
$$X_L = X_C \Rightarrow 2\pi f L = \frac{1}{2\pi f C} \Rightarrow f^2 = \frac{1}{4\pi^2 LC}$$

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(5 \times 10^{-3}\ \text{H})(0.1 \times 10^{-6}\ \text{F})}}$$
$$= \frac{1}{2\pi\sqrt{5 \times 10^{-10}}} = \frac{1}{(2\pi)(2.236 \times 10^{-5})} = \frac{10^5\ \text{Hz}}{14.05} \cong \mathbf{7.12\ kHz}$$

14.5 AVERAGE POWER AND POWER FACTOR



Demonstrating that power is delivered at every instant of a sinusoidal voltage waveform (except $v_R = 0$ V).



Power versus time for a purely resistive load.

Any portion of the power curve below the axis reveals that power is being returned to the source. The average value of the power curve occurs at a level equal to *Vm Im/2 as shown in Fig. This power level* is called the **average or real power level.**

$$P_{av} = \frac{V_m I_m}{2} = \frac{(\sqrt{2}\,V_{rms})(\sqrt{2}\,I_{rms})}{2} = \frac{2\,V_{rms} I_{rms}}{2}$$

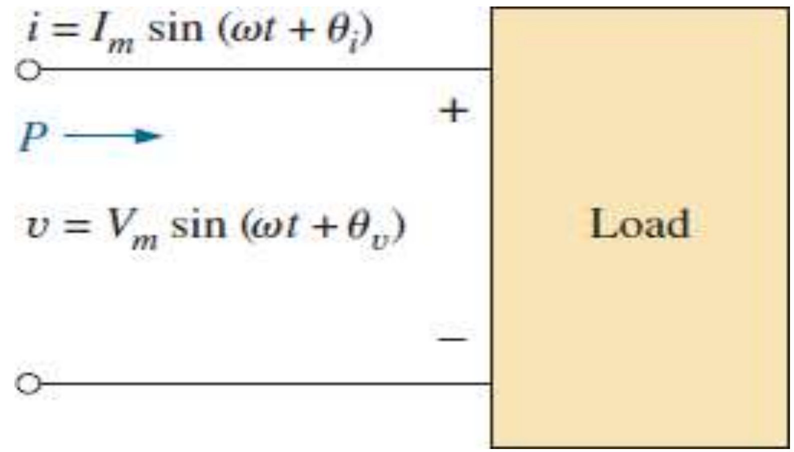
$$\boxed{P_{av} = V_{rms}\, I_{rms}}$$

If the sinusoidal voltage is applied to a network with a combination of R, L, and C components, the instantaneous equation for the power levels is more complex.

In Fig., a voltage with an initial phase angle is applied to a network with any combination of elements that results in a current with the indicated phase angle.

The power delivered at each instant of time is then defined by:

$$\begin{aligned} p = vi &= V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i) \\ &= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i) \end{aligned}$$



Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

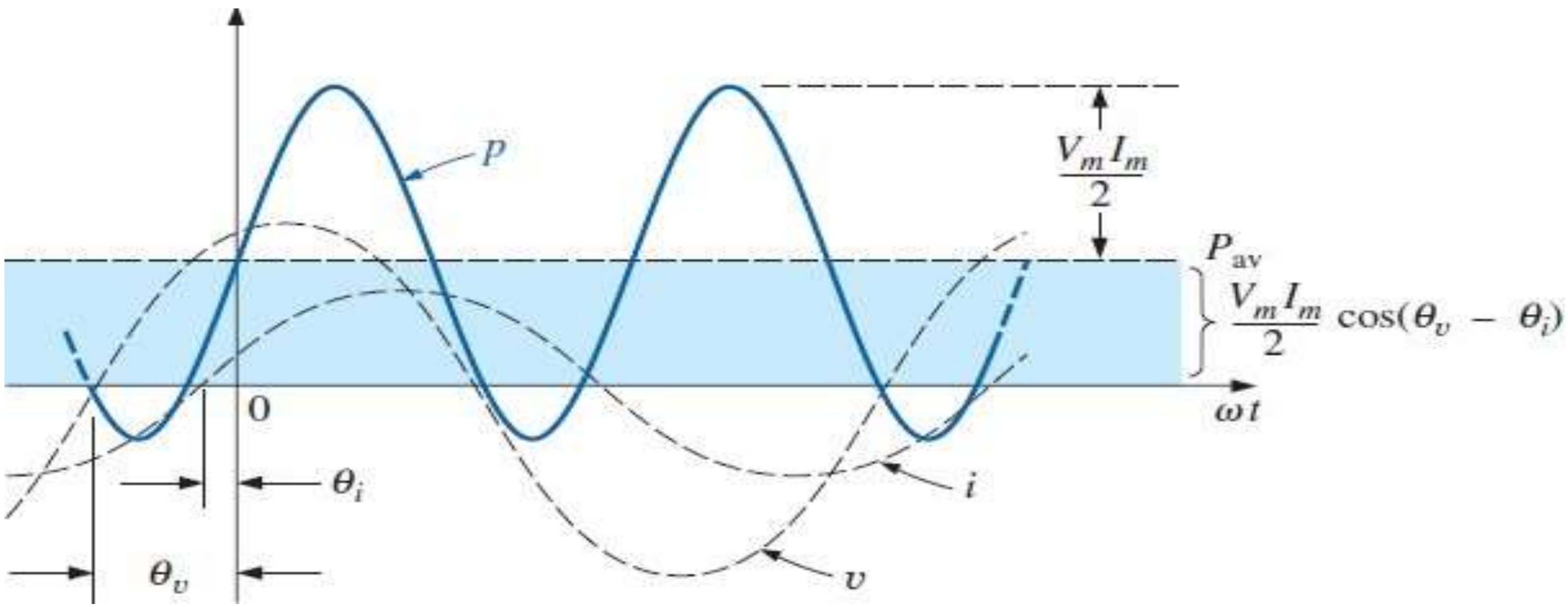
the function $\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$ becomes

$$\begin{aligned} &\sin(\omega t + \theta_v) \sin(\omega t + \theta_i) \\ &= \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2} \\ &= \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2} \end{aligned}$$

so that

$$p = \left[\overbrace{\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)}^{\text{Fixed value}}\right] - \left[\overbrace{\frac{V_m I_m}{2}\cos(2\omega t + \theta_v + \theta_i)}^{\text{Time-varying (function of } t)}\right]$$

Note that the second factor in the preceding equation is a cosine wave with an amplitude of VmIm /2 and with a frequency twice that of the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.



Since cos(-α) = cos α, *the magnitude of average power delivered is independent of whether* v *leads i or i leads v.*

Defining θ as equal to $|\theta_v - \theta_i|$, where $|\ \ |$ indicates that only the magnitude is important and the sign is immaterial, we have

$$P = \frac{V_m I_m}{2} \cos\theta \qquad \text{(watts, W)} \qquad P = V_{\text{rms}} I_{\text{rms}} \cos\theta$$

where P is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since $$V_{\text{eff}} = \frac{V_m}{\sqrt{2}} \quad \text{and} \quad I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$$

Resistor

In a purely resistive circuit, since v and i are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$P = \frac{V_m I_m}{2} = V_{rms} I_{rms} \qquad P = \frac{V_{rms}^2}{R} = I_{rms}^2 R \qquad (W)$$

Inductor

In a purely inductive circuit, since v leads i by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

Capacitor

In a purely capacitive circuit, since i leads υ by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 14.10 Find the average power dissipated in a network whose input current and voltage are the following:

$$i = 5\sin(\omega t + 40°)$$
$$\upsilon = 10\sin(\omega t + 40°)$$

Since v and i are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\text{ V})(5\text{ A})}{2} = \mathbf{25\ W}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\text{ V}}{5\text{ A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{[(0.707)(10\text{ V})]^2}{2} = \mathbf{25\ W}$$

or

$$P = I_{\text{rms}}^2 R = [(0.707)(5\text{ A})]^2(2) = \mathbf{25\ W}$$

EXAMPLE 14.11 Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100 \sin(\omega t + 40^\circ)$
$i = 20 \sin(\omega t + 70^\circ)$

b. $v = 150 \sin(\omega t - 70^\circ)$
$i = 3 \sin(\omega t - 50^\circ)$

a. $V_m = 100, \quad \theta_v = 40^\circ$
$I_m = 20\text{ A}, \quad \theta_i = 70^\circ$
$\theta = |\theta_v - \theta_i| = |40^\circ - 70^\circ| = |-30^\circ| = 30^\circ$

and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(100\text{ V})(20\text{ A})}{2} \cos(30^\circ) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\ W}$$

b. $V_m = 150\text{ V}, \quad \theta_v = -70^\circ$
$I_m = 3\text{ A}, \quad \theta_i = -50^\circ$
$\theta = |\theta_v - \theta_i| = |-70^\circ - (-50^\circ)|$
$= |-70^\circ + 50^\circ| = |-20^\circ| = 20^\circ$

and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(150\text{ V})(3\text{ A})}{2} \cos(20^\circ) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.43\ W}$$

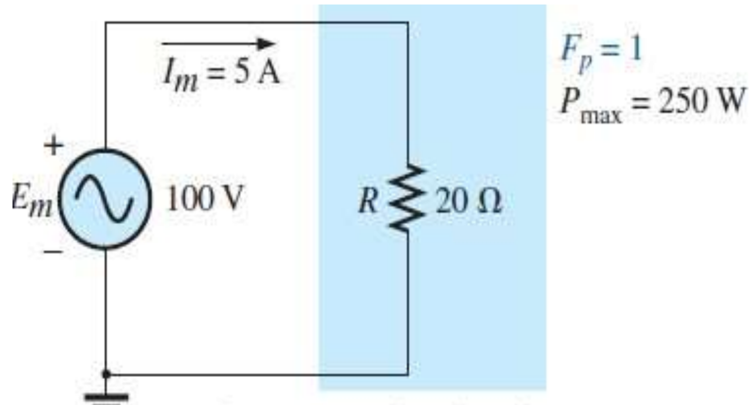
Power Factor

In the equation P = (VmIm/2)cos Θ, the factor that has significant control over the delivered power level is the cos Θ. No matter how large the voltage or current, if cos Θ = 0, the power is zero; if cos Θ = 1, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

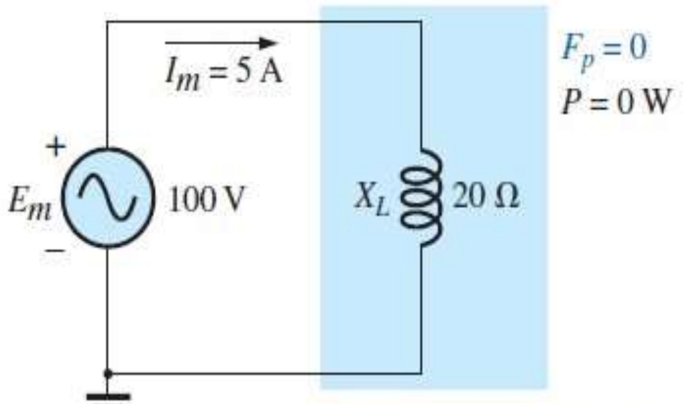
$$\text{Power factor} = F_p = \cos\theta$$

In terms of the average power and the terminal voltage and current,

$$F_p = \cos\theta = \frac{P}{V_{rms} I_{rms}}$$



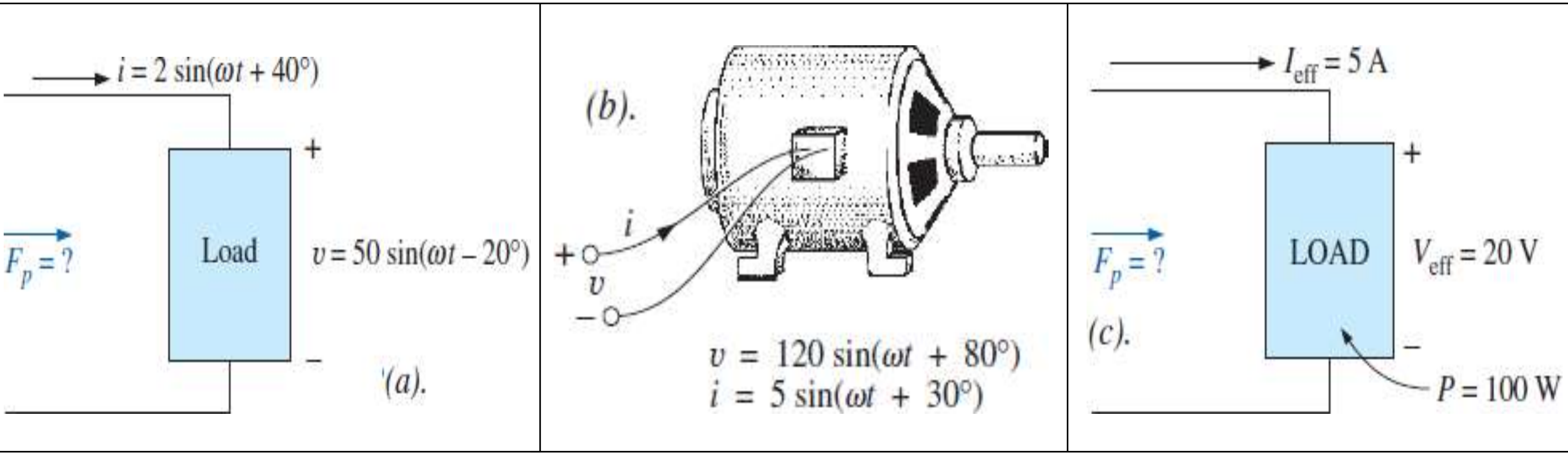
Purely resistive load with $F_p = 1$.



Purely inductive load with $F_p = 0$.

Capacitive networks have leading power factors, and inductive networks have lagging power factors.

EXAMPLE 14.12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:



a. $F_p = \cos\theta = \cos\,|40^\circ - (-20^\circ)| = \cos 60^\circ =$ **0.5 leading**

b. $F_p = \cos\theta\,|80^\circ - 30^\circ| = \cos 50^\circ =$ **0.64 lagging**

c. $F_p = \cos\theta = \dfrac{P}{V_{eff}I_{eff}} = \dfrac{100\text{ W}}{(20\text{ V})(5\text{ A})} = \dfrac{100\text{ W}}{100\text{ W}} = \mathbf{1}$

The load is resistive, and F_p is neither leading nor lagging.

THANK YOU