Lecture No. 4

Applications of the Derivatives

4.1 Extreme Values of Functions on Closed Intervals

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of optimization problems. The domains of the functions we consider are intervals or unions of separate intervals

> **DEFINITIONS** Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if
>
> $$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$
>
> and an **absolute minimum** value on D at c if
>
> $$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima.

For example, on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the function $f(x) = \cos x$ takes on an absolute **maximum value of 1** (once) and an absolute **minimum value of 0** (twice). On the same interval, the function g(x) = sin x takes on a maximum value of 1 and a minimum value of -1 (Figure 4-1)

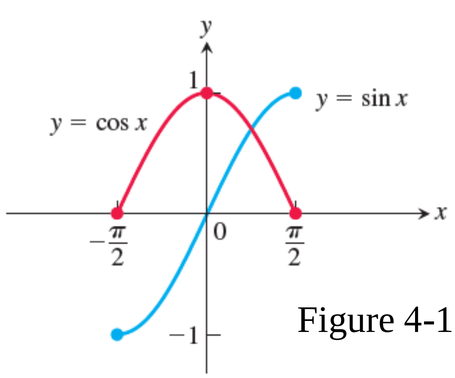
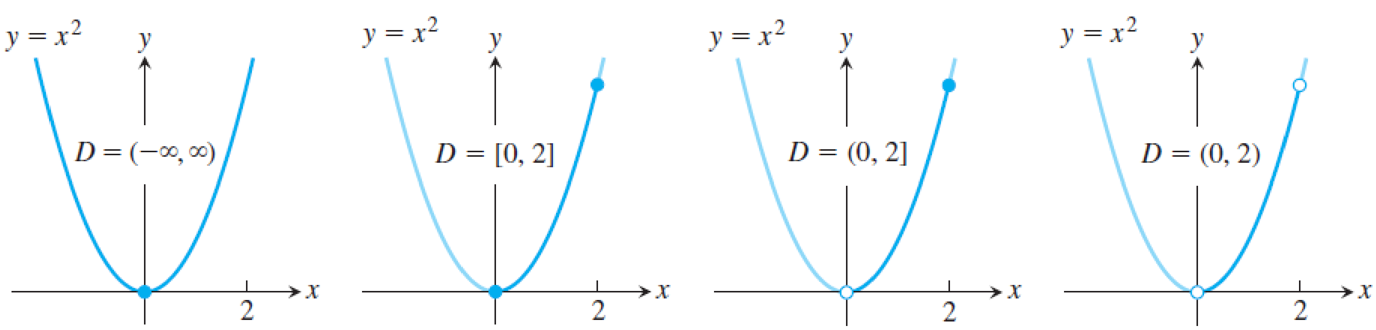


Figure 4-1

Functions defined by the same equation or formula can have **different extrema (maximum or minimum values), depending on the domain**. A function might **not have a maximum or minimum** if the domain is **unbounded** or fails to contain an endpoint. We see this in the following example.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Each function has the same defining equation, y = *x*2, but the domains vary.

Function rule	**Domain *D***	**Absolute extrema on *D***
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema



(a) abs min only (b) abs max and min (c) abs max only (d) no max or min

Figure 4-2

Some of the functions in Example 1 **do not have a maximum or a minimum value.** The following theorem asserts that a function which is continuous over (or on) a finite closed interval 3a, b4 has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

THEOREM 1—The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system and we will not give it here. Figure 4.3 b illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b\ ]$As we observed for the function $y = cos\ x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval. The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are essential. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. The function $y = x$ over $(-\infty, \infty)$ shows that neither extreme value need exist on an infinite interval. Figure 4.4 shows that the continuity requirement cannot be omitted. Figure 4.3 a

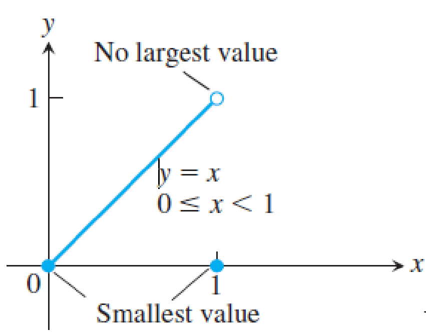
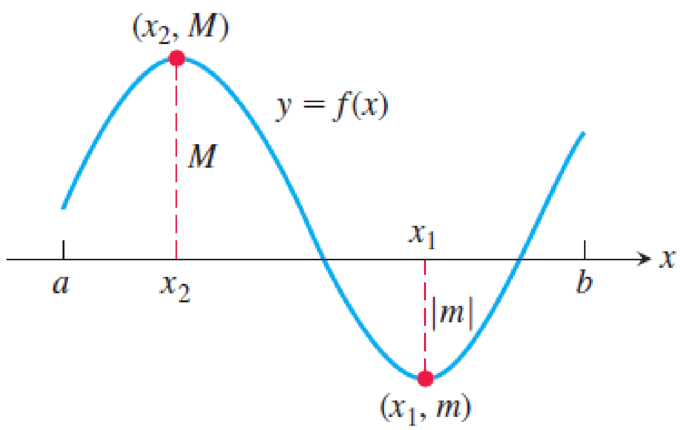
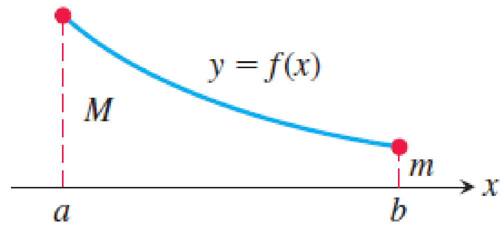


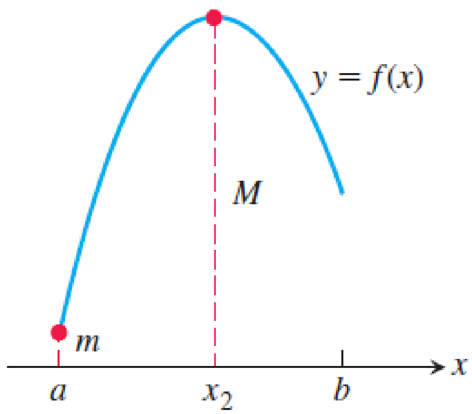
Figure 4-3 a



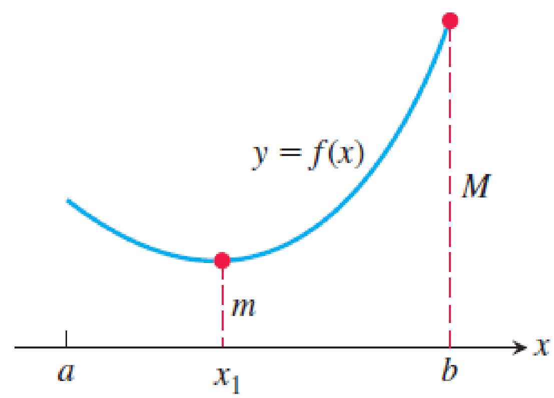
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



Maximum at interior point,
minimum at endpoint



Minimum at interior point,
maximum at endpoint

Figure 4-4 b

4.2 Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2—The First Derivative Theorem for Local Extreme Values
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1.$

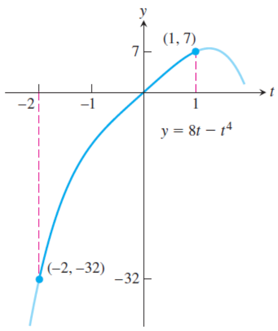
The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \qquad \text{or} \qquad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See Figure 4.8. ∎



EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

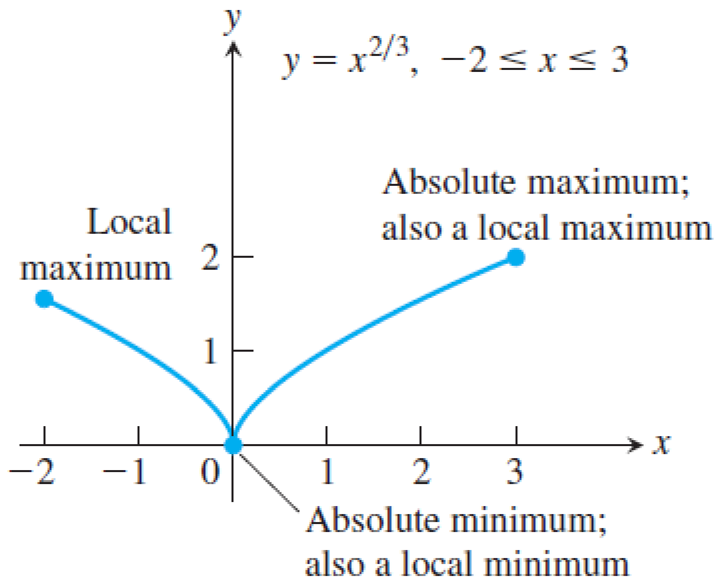


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

HOMEWORK

Absolute Extrema on Finite Closed Intervals

In Exercises 21–36, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

22. $f(x) = -x - 4, \quad -4 \le x \le 1$

23. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

24. $f(x) = 4 - x^3, \quad -2 \le x \le 1$

25. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

26. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

27. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

28. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

29. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

30. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

31. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

32. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

33. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$

34. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

35. $f(t) = 2 - |t|, \quad -1 \le t \le 3$