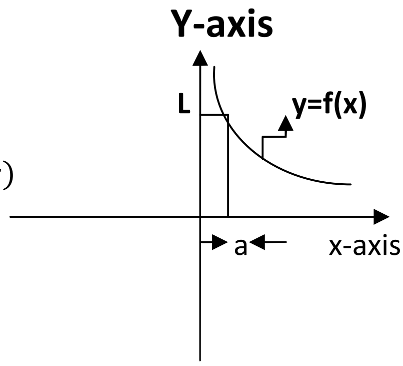
Lecture No. 2
Limits and Continuity

LIMITS OF FUNCTION:



Definitions:

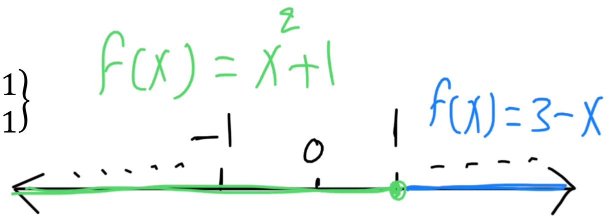
$\boldsymbol{\lim_{x\to a} f(x) = L}$ *Mean that when a value of* (x) *close to* $(a) f(x)$ *approaches the limiting value* (l).

$\boldsymbol{\lim_{x\to a^+} f(x) = L}$ *Mean that* (x) *approaches* (a) *from the right.*

$\boldsymbol{\lim_{x\to a^-} f(x) = L}$ *Mean that* (x) *approaches* (a) *from the left.*

Note: $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x) = L$ *we say that* $\lim_{x\to a} f(x)$ *exist. otherwise the limit doesn't exist.*

Ex: Find $\lim_{x\to 1} f(x)$ *where* $f(x) = \begin{cases} x^2+1 & x \le 1 \\ 3-x & x > 1 \end{cases}$

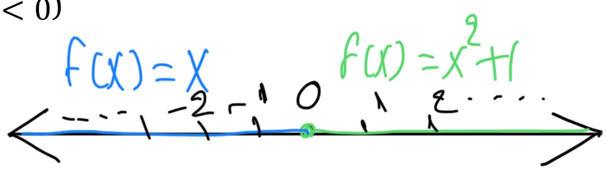


Sol: $\lim_{x\to 1^-} f(x) = 1^2 + 1 = 2$

$\lim_{x\to 1^+} f(x) = 3 - 1 = 2$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2,$ *so the limit exist.*

Ex: Find $\lim_{x\to 0} f(x)$ *where* $f(x) = \begin{cases} x^2+1 & x \ge 0 \\ x & x < 0 \end{cases}$



$\lim_{x\to 0^-} f(x) = x = 0$

$\lim_{x\to 0^+} f(x) = 0 + 1 = 1$

$\therefore \quad \lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ *does't exist.*

Note that **– A function *F (t)* has a limit at point *C* if and only if the right hand and the left hand limits at *C* exist and equal. In symbols:**

$\lim_{t\to C} F(t) = L \Leftrightarrow \lim_{t\to C^+} F(t)\ L \text{ and } \lim_{t\to C^-} F(t)\ L$

Properties of Limits: خصائص الغايات

Let: $\lim_{x\to a} f(x) = L1$

$\lim_{x\to a} g(x) = L2$

& K is a constant, then:

1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L1 + L2$

2) $\lim_{x\to a}[f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$

3) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ but $\lim_{x\to a} g(x) \neq 0$

4) $\lim_{x\to a} K * f(x) = K * \lim_{x\to a} f(x)$

5) $\lim_{x\to a} K = K$

6) $\lim_{x\to a} x = a$

7) $\lim_{x\to\infty} \frac{1}{x} = 0$ & $\lim_{x\to\infty} \sqrt{x} = \infty$

8) $\lim_{x\to 0+} \frac{1}{x} = +\infty$ & $\lim_{x\to 0-} \frac{1}{x} = -\infty$ but $\lim_{x\to 0} \frac{x}{1} = 0$

9) $\lim_{x\to 0} sinx = 0$ & $\lim_{x\to 0} cosx = 1$ & $\lim_{x\to 0} tanx = 0$

10) $\lim_{x\to 0} \frac{sinx}{x} = 1$ & $\lim_{x\to 0} \frac{x}{sinx} = 1$

11) $\lim_{x\to 0} \frac{tanx}{x} = 1$ & $\lim_{x\to 0} \frac{x}{tanx} = 1$

12) $\lim_{x\to a} \sin\left(\frac{x^2}{\pi+x}\right) = \sin\left(\lim_{x\to a} \frac{x^2}{\pi+x}\right)$ Note: sine or cosine or any trigonometric functions is the same

13) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ & $\lim_{x\to a} \frac{1}{x^n} = [\lim_{x\to a} \frac{1}{x}]^n$

Note:

Undefined expression in limits:

$\frac{0}{0}, \frac{\infty}{\infty}, 0 * \infty, \infty - \infty$ *but we can say* $\infty + \infty = \infty$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example :

$$\lim_{x \to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = 0$$

Eliminating Zero Denominators Algebraically :

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c.

Examples :

1. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x}$

$= \frac{1+2}{1} = 3$

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$$(a-b)(a+b) = a^2 - b^2$$

2. $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} \times \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$

$\lim_{x\to 0} \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$

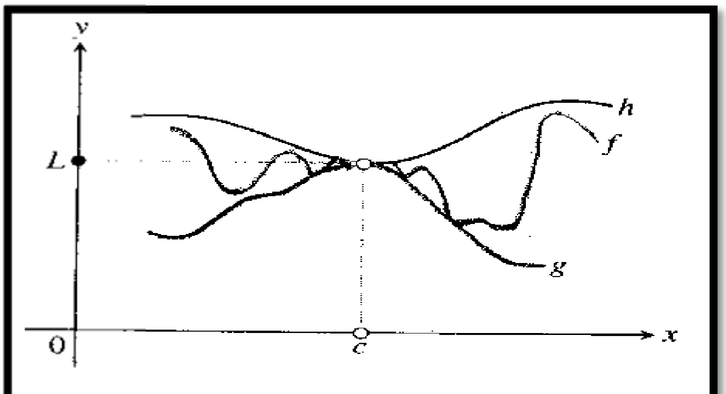
$\lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{\sqrt{0+100}+10} = \frac{1}{20} = 0.05$

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.



The graph of f is sandwiched between the graphs of g and h.

Example :

$$1-\frac{x^2}{4} \leqslant u(x) \leqslant 1+\frac{x^2}{2} \quad \text{for all } x \neq c$$

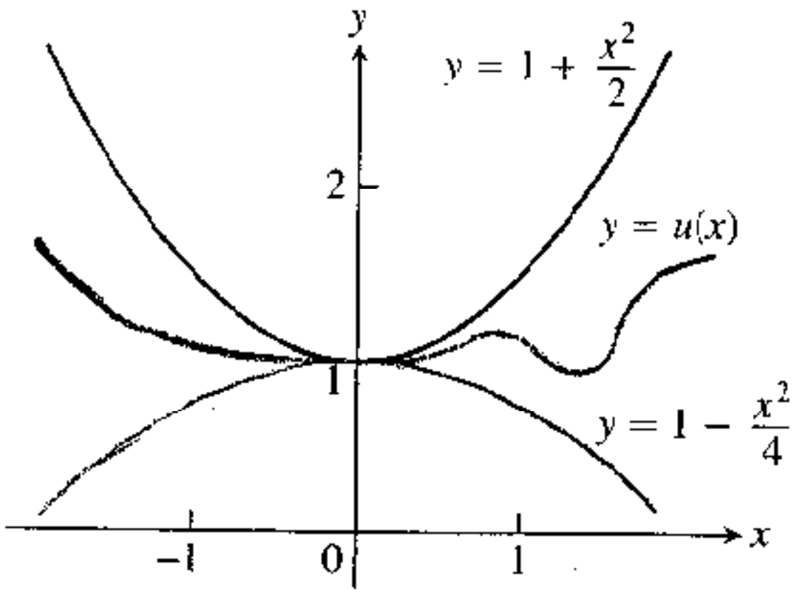
find $\lim_{x\to 0} u(x)$.

Solution :

$$\lim_{x\to 0}\left(1-\frac{x^2}{4}\right) = \left(1-\frac{0}{4}\right) = 1$$

$$\lim_{x\to 0}\left(1+\frac{x^2}{2}\right) = \left(1+\frac{0}{2}\right) = 1$$

$$\lim_{x\to 0} u(x) = 1$$



THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The basic facts to be verified by applying the formal definition which are: $\lim_{x\to\pm\infty} k = k$ *and* $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

THEOREM **Limit Laws as $x \to \pm\infty$**

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Examples:

1. $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}=5+0=5$

2. $\lim_{x\to\infty}\frac{\pi\sqrt{3}}{x^2}=\lim_{x\to\infty}\pi\sqrt{3}\frac{1}{x}\cdot\frac{1}{x}=\lim_{x\to\infty}\pi\sqrt{3}\cdot\lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1}{x}=\pi\sqrt{3}\cdot 0.0=0$

Limits at Infinity of Rational Functions:

Example: (***Numerator*** & *Denominator of same De**gree***)

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+\left(\frac{8}{x}\right)-\left(\frac{3}{x^2}\right)}{3+\left(\frac{2}{x^2}\right)}=\frac{5+0-0}{3+0}=\frac{5}{3}$$

Example: (***Degree of Numerator less than Degree of Denominator***)

$$\lim_{x\to\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to\infty}\frac{\left(\frac{11}{x^2}\right)+\left(\frac{2}{x^3}\right)}{2-\left(\frac{1}{x^3}\right)}=\frac{0+0}{2-0}=0$$

Example: (*Degree of Numerator greater than degree of Denominator*)

$$\lim_{x\to-\infty}\frac{2x^2-3}{7x+4}=\lim_{x\to-\infty}\frac{(2x)-\left(\frac{3}{x}\right)}{7+\left(\frac{4}{x}\right)}=\frac{2(-\infty)-0}{7+0}=-\infty$$

Examples:

1) $$\lim_{x\to\infty}x-\sqrt{x^2+1}=\lim_{x\to\infty}x-\sqrt{x^2+1}\times\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{x^2-(x^2+1)}{x+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{-1}{x+\sqrt{x^2+1}}=-\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}=\frac{0}{1+\sqrt{1+0}}=-\frac{0}{2}=0$$

2) $$\lim_{x\to\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}=\lim_{x\to\infty}\frac{1-\left(\frac{x^{\frac{1}{5}}}{x^{\frac{1}{3}}}\right)}{1+\left(\frac{x^{\frac{1}{5}}}{x^{\frac{1}{3}}}\right)}=\lim_{x\to\infty}\frac{1-\left(\frac{1}{x^{\frac{2}{15}}}\right)}{1+\left(\frac{1}{x^{\frac{2}{15}}}\right)}=\frac{1-0}{1+0}=\frac{1}{1}=1$$

3) $$\lim_{x\to\infty}\sqrt{\frac{2+3x}{1+5x}}=\sqrt{\lim_{x\to\infty}\frac{2+3x}{1+5x}}=\sqrt{\lim_{x\to\infty}\frac{\frac{2}{x}+\frac{3x}{x}}{\frac{1}{x}+\frac{5x}{x}}}=\sqrt{\frac{3}{5}}$$

4) $\lim_{x\to 2}\dfrac{x^6-64}{x-2}$ $\qquad x^5+2x^4+4x^3+8x^2+16x+32$

$$\begin{array}{l|l} x^6-64 & x-2 \\ \hline \mp x^6 \pm 2x^5 \quad \Leftarrow x^5(x-2) & \\ 2x^5-64 & \\ \mp 2x^5 \pm 4x^4 \quad \Leftarrow 2x^4(x-2) & \\ 4x^4-64 & \\ \mp 4x^4 \pm 8x^3 \quad \Leftarrow 4x^3(x-2) & \\ 8x^3-64 & \\ \mp 8x^3 \pm 16x^2 \quad \Leftarrow 8x^2(x-2) & \\ 16x^2-64 & \\ \mp 16x^2 \pm 32x \quad \Leftarrow 16x(x-2) & \\ 32x-64 & \\ \mp 32x \pm 64 \quad \Leftarrow 32(x-2) & \\ 0 \quad 0 & \end{array}$$

$$\lim_{x\to 2}\frac{x^6-64}{x-2}=\lim_{x\to 2}x^5+2x^4+4x^3+8x^2+16x+32$$
$$=2^5+2(2^4)+4(2^3)+8(2^2)+16(2)+32=192$$

Note :

$f(x)=\dfrac{x^2-3}{2x-4}$

$\dfrac{x^{\not 2\,1}}{2\not x}=\dfrac{x}{2}$

$$\begin{array}{l|l} \quad \frac{x}{2}+1 & \\ x^2-3 & 2x-4 \\ \hline \mp x^2 \pm 2x \quad \Leftarrow \frac{x}{2}(2x-4) & \\ 2x-3 & \\ \mp 2x \pm 4 \quad \Leftarrow 1(2x-4) & \\ 1 & \end{array}$$

$f(x)=\dfrac{x^2-3}{2x-4}=\dfrac{x}{2}+1+\dfrac{1}{2x-4}$

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***Continuity*:**

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

If a function f is not continuous at a point c, we say that f is discontinuous at c and c is a point of discontinuity of f.

Note that c need not be in the domain of f.

Continuity Test

A function $f(x)$ is continuous at an interior point of its domain $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example: The function $y = \frac{1}{x}$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$.

THEOREM Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c) = (f+g)(c)$$

This shows that $f+g$ *is continuous.*

THEOREM: Composite of Continuous Functions

If f *is continuous at* c *and* g *is continuous at* c*, then the composite* $g \, o \, f$ *is continuous at* c.

The following types of functions are continuous at every point in their domains:

1 − *Polynomials.*

2 − *Rational functions: They have points of discontinuity at the zero of their denominators.*

3 − *Root functions:* $\left(y = \sqrt[n]{x}\right.$ *, n a positive integer greater than* $\left.1\right)$.

4 − *Trigonometric functions.*

5 − *Inverse trigonometric functions.*

6 − *Exponential functions.*

7 − *Logarithmic functions.*

Note: The inverse function of any continuous function is continuous.

Examples:

1) $f(x) = \begin{Bmatrix} \frac{2x^2+x-3}{x-1} & x \neq 1 \\ 2 & x = 1 \end{Bmatrix}$

$1 - f(1) = 2$

$2 - \lim_{x\to 1} \frac{2x^2+x-3}{x-1} = \lim_{x\to 1} \frac{(2x+3)(x-1)}{(x-1)} = 2(1)+3 = 5$

$3 - \lim_{x\to 1} f(x) \neq f(1)$

$\therefore f(x)$ *discontinuous at* $x = 1$.

2) $f(x) = \begin{Bmatrix} 3+x & x \leq 1 \\ 3-x & x > 1 \end{Bmatrix}$

$1 - f(1) = 3+1 = 4$

$2 - \lim_{x\to 1^-} 3+x = 3+1 = 4$

$3 - \lim_{x\to 1^+} 3-x = 3-1 = 2 \qquad \therefore \lim_{x\to 1^-} 3+x \neq \lim_{x\to 1^+} 3-x$

$\therefore f(x)$ *discontinuous at* $x = 1.$

3) $f(x) = \begin{cases} \frac{1}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

$f(2) = 3$ & $\lim_{x\to 2} \frac{1}{x-2} = \frac{1}{0} = \infty$ $\quad \therefore$ *no limit* , $f(x)$ *discontinuous.*

4) $f(x) = \begin{cases} \frac{x^2-16}{x-4} & x \neq 4 \\ 9 & x = 4 \end{cases}$

$f(4) = 9$, $\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)} = 8,\ f(4) \neq \lim_{x\to 4} f(x)$

$\therefore f(x)$ *discontinuous at* $x = 4.$

5) $f(x) = \begin{cases} |x-3| & x \neq 3 \\ 2 & x = 3 \end{cases}$

$f(3) = 2$, $\quad \lim_{x\to 3^+} (x-3) = 3 - 3 = 0, \quad \lim_{x\to 3^-} -(x-3) = 0$

$$\lim_{x\to 3^+} (x-3) = \lim_{x\to 3^-} -(x-3) = 0$$

$$but\ \ f(3) \neq \lim_{x\to 3} f(x)$$

$\therefore f(x)$*discontinuous at* $x = 3.$