اسم المادة : رياضيات-1
اسم التدريسي : د حسين كاظم حلواص و م.م زين العابدين كريم
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السنة الدراسية : 2024-2023

عنوان المحاضرة:

Hyperbolic Functions and It's Derivatives, Inverse Hyperbolic Functions and It's Derivatives

الدوال الزائدية ومشتقاتها، معكوس الدوال الزائدية ومشتقاتها

Hyperbolic Functions الدوال الزائدية

- Functions which are associated with the geometry of a hyperbola are called hyperbolic fnns.
- Applications of hyperbolic fnns are:-
  1. Transmission line theory نظرية خط النقل
  2. Catenary lines الخطوط الكاتينارية

Hyperbolic fnns are:-

1. Hyperbolic sine of x ⟶ pronounced as "shine x" تلفظ بـ "شاين آكس"

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

2. Hyperbolic cosine of x ⟶ pron. as "Kosh x" تلفظ بـ "كوش آكس"

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

3. Hyperbolic tangent of x ⟶ pron. as "than x" تلفظ بـ "ثان آكس"

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

4. Hyperbolic cosecant of x ⟶ pron. as "cosheck x" تلفظ بـ "كوشك آكس"

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

5. Hyperbolic secant of x ⟶ pron. as "shec x" تلفظ بـ (شك آكس)

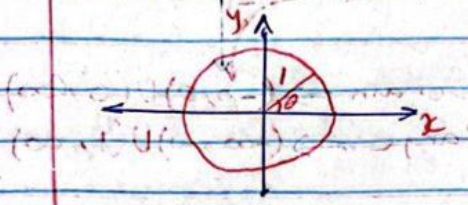
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

6. Hyperbolic cotangent of x ⟶ pron. as "keth x" تلفظ بـ (كوث آكس)

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Trigonometric Fnns. Vs hyperbolic Fnns
الدوال المثلثية مقابل الدوال الزائدية

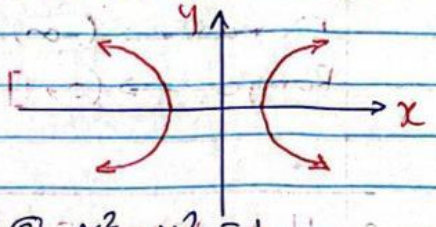
Trigonometric Fnns
① Based on the unit circle.



② $x^2+y^2=1$

③ $\cos^2x+\sin^2x=1$

Hyperbolic Fnns
① based on the hyperbola.

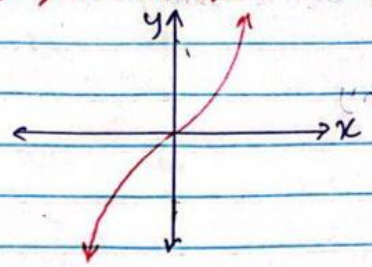


② $x^2-y^2=1$

③ $\cosh^2x-\sinh^2x=1$

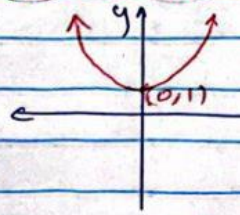
Graph of Hyperbolic Fnns.

① $Y=\sinh x$



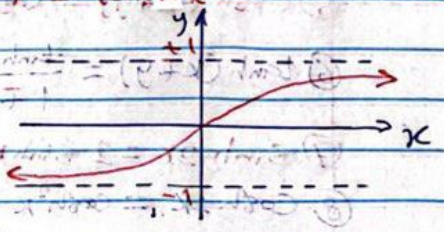
Domain → $(-\infty,+\infty)$
Range → $(-\infty,+\infty)$

② $Y=\cosh x$



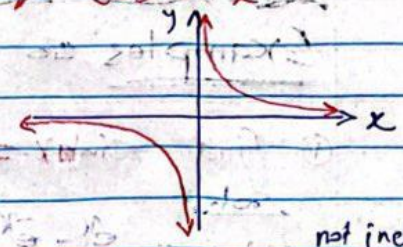
Domain → $(-\infty,+\infty)$
Range → $[1,+\infty)$

③ $Y=\tanh x$



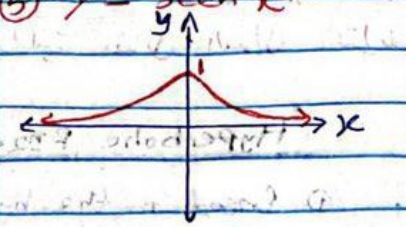
Domain → $(-\infty,+\infty)$
Range → $(-1,1)$

④ $Y=\operatorname{cosech} x$



Domain → $(-\infty,0)\cup(0,+\infty)$ (not included)
Range → $(-\infty,0)\cup(0,+\infty)$ (not included)

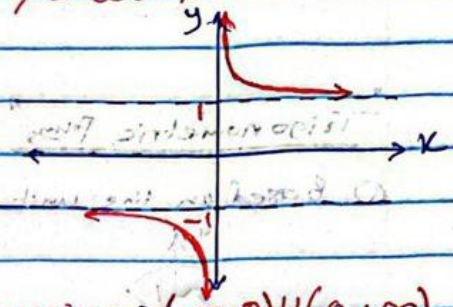
⑤ $y = \operatorname{sech} x$



Domain → $(-\infty, +\infty)$

Range → $(0, 1]$

⑥ $y = \coth x$



Domain → $(-\infty, 0) \cup (0, \infty)$

Range → $(-\infty, -1) \cup (1, \infty)$

Hyperbolic f-ns Identities ex

① $\cosh^2 x - \sinh^2 x = 1$

② $1 - \tanh^2 x = \operatorname{sech}^2 x$

③ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

④ $\sinh(x \mp y) = \sinh x \cosh y \mp \cosh x \sinh y$

⑤ $\cosh(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y$

⑥ $\tanh(x \mp y) = \dfrac{\tanh x \mp \tanh y}{1 \mp \tanh x \tanh y}$

⑦ $\sinh 2x = 2 \sinh x \cosh x$

⑧ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x$

⑨ $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Examples ex

① Find $\sinh x = 3$?

Sol:

$\sinh x = \dfrac{e^x - e^{-x}}{2} = 3 \rightarrow e^x - e^{-x} = 6$

$\rightarrow (e^x - e^{-x} - 6 = 0) \times e^x$

$(e^x)^2 - e^{-x} e^x - 6e^x = 0 \rightarrow (e^x)^2 - e^{(-x+x)} - 6e^x = 0$

$e^{-x+x} = e^0 = 1$

$\therefore (e^x)^2 - 6e^x - 1 = 0$

$\therefore e^x = \frac{-(-6) \mp \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \mp \sqrt{36+4}}{2} = \frac{6 \mp \sqrt{40}}{2}$

$\therefore e^x = \frac{6 \mp 6.3246}{2} \longrightarrow \therefore \boxed{e^x = 6.1623} \text{ or } \boxed{e^x = -0.1623}$

② $y = 40 \cosh \frac{x}{40}$; $x = 25$

Sol:

$y = 40 \cosh \frac{25}{40} = 40 \cosh(0.625) = \frac{40}{2}\left(e^{0.625} + e^{-0.625}\right)$

$= 20(1.8682 + 0.5353) = \boxed{48.07}$

③ $y = 40 \cosh \frac{x}{40}$; $y = 54.3$

Sol:

$54.3 = 40 \cosh \frac{x}{40} \longrightarrow \cosh \frac{x}{40} = \frac{54.3}{40} = 1.3575$

$\frac{e^{x/40} + e^{-x/40}}{2} = 1.3575 \longrightarrow e^{x/40} + e^{-x/40} = 2.715 \longrightarrow$

$\left(e^{x/40} + e^{-x/40} - 2.715 = 0\right) * e^{x/40}$

$(e^{x/40})^2 + 1 - 2.715 e^{x/40} = 0 \longrightarrow (e^{x/40})^2 - 2.715 e^{x/40} + 1 = 0$

$\therefore e^{x/40} = \frac{-(-2.715) \mp \sqrt{(-2.715)^2 - 4(1)(1)}}{2(1)} = \frac{2.715 \mp \sqrt{3.3712}}{2}$

$\therefore \left\{e^{x/40} = 2.2756 \text{ or } 0.43945\right\} * \ln$

$\frac{x}{40} = \ln(2.2756) \text{ or } \frac{x}{40} = \ln(0.43945)$

$\therefore \boxed{x = \mp 32.89}$

The Derivative of Hyperbolic Functions
مشتقة الدوال الزائدية

① $\frac{d}{dx}\sinh u = \cosh u * u'$

② $\frac{d}{dx}\cosh u = \sinh u * u'$

③ $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u * u'$

④ $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u * u'$

⑤ $\frac{d}{dx}\operatorname{cosech} u = -\operatorname{cosech} u \coth u * u'$

⑥ $\frac{d}{dx}\coth u = -\operatorname{cosech}^2 u * u'$

Examples

① Find the derivative of $y = 4\sinh 2x - \frac{3}{7}\cosh 3x$

Sol:

$$\frac{dy}{dx} = 4(\cosh 2x * 2) - \frac{3}{7}(\sinh 3x * 3)$$
$$= \boxed{8\cosh 2x - \frac{9}{7}\sinh 3x}$$

② Derive $y = 5\tanh\frac{x}{2} - 2\coth 4x$

Sol:

$$\frac{dy}{dx} = 5(\operatorname{sech}^2\frac{x}{2} * \frac{1}{2}) - 2(-\operatorname{cosech}^2 4x * 4)$$
$$= \boxed{\frac{5}{2}\operatorname{sech}^2\frac{x}{2} + 8\operatorname{cosech}^2 4x}$$

Inverse Hyperbolic Functions :- الدوال الزائدية العكسية

Inverse hyperbolic fns are denoted by using the -1 notation.

If $y = \sinh x$, then $x = \sinh^{-1} y$, and

If $y = \operatorname{sech} x$, then $x = \operatorname{sech}^{-1} y$, and so on.

① $y = \sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right]$, Domain $= [1, \infty)$

② $y = \cosh^{-1} x = \ln\left[x + \sqrt{x^2-1}\right]$, Domain $= (-\infty, \infty)$

③ $y = \tanh^{-1} x = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right|$, Domain $= (-1, 1)$

④ $y = \operatorname{cosech}^{-1} x = \ln\left|\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right|$, Dom. $= (-\infty, 0) \cup (0, \infty)$

⑤ $y = \operatorname{sech}^{-1} x = \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|$, Domain $= (0, 1]$

⑥ $y = \coth^{-1} x = \frac{1}{2} \ln\left|\frac{x+1}{x-1}\right|$, Domain $= (-\infty, -1) \cup (1, \infty)$

Examples

① Evaluate $\sinh^{-1} 2$

Soln

$$\sinh^{-1} 2 = \ln\left[2 + \sqrt{2^2+1}\right] = \ln\left[2 + \sqrt{5}\right]$$
$$= \ln(4.2361) = \boxed{1.4436}$$

② Evaluate $\cosh^{-1} 1.4$

Soln

$$\cosh^{-1} 1.4 = \ln\left[1.4 + \sqrt{1.4^2 - 1}\right] = \ln\left[1.4 + 0.979\right]$$
$$= \ln(2.379) = \boxed{0.867}$$

The Derivative of Inverse Hyperbolic Functions

مشتقة معكوس الدوال الزائدية

① $\frac{d}{dx}\sinh^{-1}u = \frac{u'}{\sqrt{u^2+1}}$

② $\frac{d}{dx}\cosh^{-1}u = \frac{u'}{\sqrt{u^2-1}}$

③ $\frac{d}{dx}\tanh^{-1}u = \frac{u'}{1-u^2}$

④ $\frac{d}{dx}\operatorname{cosech}^{-1}u = \frac{-u'}{|u|\sqrt{1+u^2}}$

⑤ $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-u'}{u\sqrt{1-u^2}}$

⑥ $\frac{d}{dx}\coth^{-1}u = \frac{u'}{1-u^2}$

Examples :

① Find the derivative of $y = \sinh^{-1}(4x)$

Sol.

$$\frac{dy}{dx} = \frac{4}{\sqrt{(4x)^2+1}} = \boxed{\frac{4}{\sqrt{16x^2+1}}}$$

② Derive $y = \cosh^{-1}(x^3)$

Sol.

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{(x^3)^2-1}} = \boxed{\frac{3x^2}{\sqrt{x^6-1}}}$$

③ Derive $y = \tanh^{-1}(\sqrt{x})$

Sol.

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}}}{1-(\sqrt{x})^2} = \frac{\frac{1}{2\sqrt{x}}}{1+x} = \boxed{\frac{1}{2\sqrt{x}(1+x)}}$$

③ Derive $y = (\text{cosech}^{-1}(x))^3$

Soln

$$\frac{dy}{dx} = 3(\text{cosech}^{-1}(x))^2 * \frac{-1}{|x|\sqrt{1+x^2}} = \boxed{\frac{-3(\text{cosech}^{-1}(x))^2}{|x|\sqrt{1+x^2}}}$$

④ Derive $y = 6x * \sinh^{-1}(3x) - 2\sqrt{1+9x^2}$

Soln

$$\frac{dy}{dx} = 6x * \frac{3}{\sqrt{(3x)^2+1}} + \sinh^{-1}(3x) * 6 - \left[2 * \frac{1}{2} * \frac{9*2x}{\sqrt{1+9x^2}}\right]$$

$$= \frac{18x}{\sqrt{9x^2+1}} + 6\sinh^{-1}(3x) - \frac{18x}{\sqrt{9x^2+1}}$$

$$= \boxed{6\sinh^{-1}(3x)}$$

نهاية محاضرة "Hyperbolic Functions and It's Derivatives, Inverse
Hyperbolic Functions and It's Derivatives
الدوال الزائدية ومشتقاتها، معكوس الدوال الزائدية ومشتقاتها"