1.

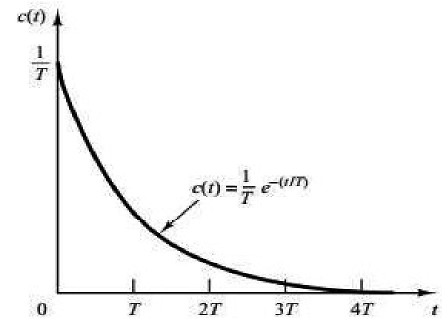
Unit-Impulse Response of First-Order Systems. For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1}$$

The inverse Laplace transform of Equation (5–7) gives

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0$$

The response curve given by Equation (5–8) is shown in Figure 5–4.



Post- Test

This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

The peak time t_p is specified as 1 sec; therefore, from Equation (5–20),

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then K_h is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Rise time t_r: From Equation (5–19), the rise time t_r is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$$

Thus, t_r is

$$t_r = 0.65 \text{ sec}$$

Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$

Ministry of high Education and Scientific Research

Middle Technical University

Electrical Engineering Technical College

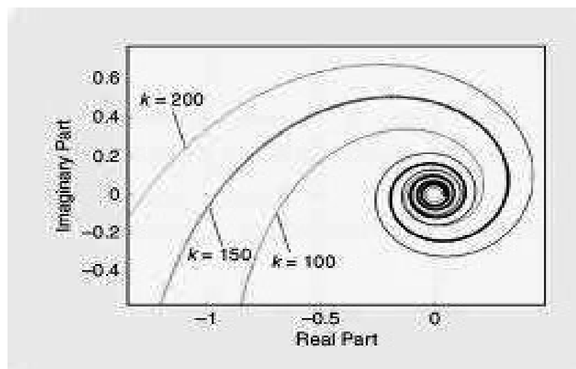
Medical Instrumentation Engineering Techniques Department

- ***Lecture 5 (10th and 11th Weeks)***

Stability Analysis; Routh, Nyquist.

Table. I Routh table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$	$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$	b_3	b_4	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$	$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	a_0				



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** A Control Systems Engineer is responsible for designing, developing, and implementing solutions that control dynamic systems.

c. **Central Ideas:** Control Systems Engineering is the engineering approach taken to understand how the process can be managed by automation devices and to implement such into operation.

d. **Objectives:** After completing this lecture, the student will be able to:

1. Define the stability in control system.
2. Describe the Routh criteria.
3. Describe the Nyquist method.

2. Pre-Test:

1. Which of the following is the best method for determining the stability and transient response?

(A) Root locus (B) Bode plot (C) Nyquist plot (D) None of the above

Answer: Option A

2. Technique gives quick transient and stability response

(A) Root locus (B) Bode (C) Nyquist (D) Nichols

Answer: Option A

Note: Check your answers in "Answer Keys" in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

3. Theory:

3.1 Stability

For nonlinear and time-varying systems, the study of stability is a complex and often difficult subject. In this section, we will consider only LTI systems for which we have the following condition for stability:

An LTI system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (that is, they are all in the left hand s-plane) and is unstable otherwise.

A system is stable if its initial conditions decay to zero and is unstable Stable system if they diverge. As just stated, an LTI (constant parameter) system is stable if all the poles of the system are strictly inside the left half s-plane [that is, all the poles have negative real parts ($s = -\sigma + j\omega$, $\sigma > 0$)]. If any pole of the system is in the right half s-plane (that is, has a positive real part, Unstable system $s = -\sigma + j\omega$, $\sigma < 0$), then the system is unstable, as shown in Fig. 3.. With any simple pole on the $j\omega$ axis ($\sigma = 0$), small initial conditions will persist. For any other pole with $\sigma = 0$, oscillatory motion will persist. Therefore, a system is stable if its transient response decays and unstable if it does not. Figure 3 shows the time response of a system due to its pole locations.

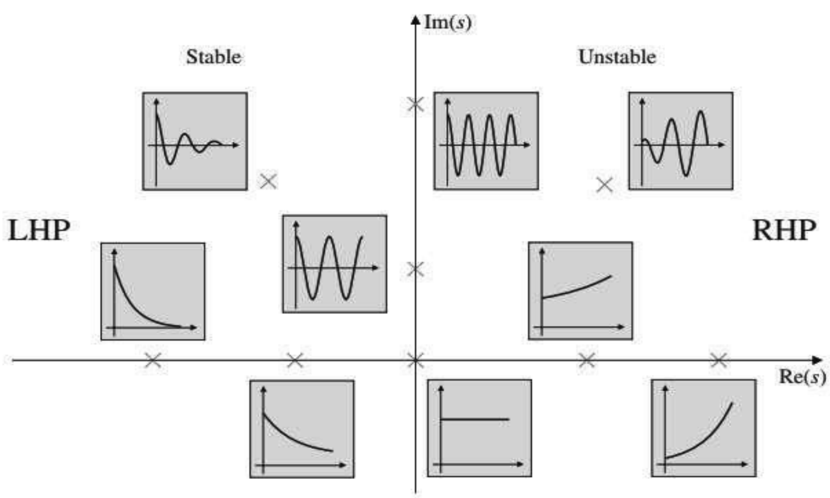


Fig 3: Time functions associated with points in the s-plane (LHP, left half-plane; RHP, right half-plane)

3.2 Routh's Stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation. The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

   where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further. Note that all the coefficients must be positive. This is a necessary condition, as may be seen from the following argument: A polynomial in s having real coefficients can always be factored into linear and quadratic factors, such as $(s + a)$ and $(s^2 + bs + c)$, where a, b, and c are real. The linear factors yield real roots and the quadratic factors yield complex-conjugate roots of the polynomial. The factor $(s^2 + bs + c)$ yields roots having negative real parts only if b and c are both positive. For all roots to have negative real parts, the constants a, b, c, and so on, in all factors must be positive. The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients. It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. The necessary but not sufficient condition for stability is that the coefficients of Equation (5–61) all be present and all have a positive sign. (If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.)

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
.	.	.			
.	.	.			
.	.	.			
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

The process of forming rows continues until we run out of elements. (The total number of rows is $n + 1$.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

.

.

.

EXAMPLE Let us apply Routh's stability criterion to the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers. The array of coefficients become

s^3	a_0	a_2
s^2	a_1	a_3
s^1	$\frac{a_1 a_2 - a_0 a_3}{a_1}$	
s^0	a_3	

The condition that all roots have negative real parts is given by

$$a_1 a_2 > a_0 a_3$$

EXAMPLE Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

s^4	1	3	5	s^4	1	3	5	
s^3	2	4	0	s^3	~~2~~	~~4~~	~~0~~	The second row is divided
					1	2	0	by 2.
s^2	1	5		s^2	1	5		
s^1	-6			s^1	-3			
s^0	5			s^0	5			

In this example, the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real parts. Note that the result is unchanged when the coefficients of any row are multiplied or divided by a positive number in order to simplify the computation.

Special Cases. If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0$$

The array of coefficients is

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (5–62) has two roots at $s = \pm j$.

If, however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2 = (s-1)^2(s+2) = 0$$

the array of coefficients is

$$\begin{array}{llcc} \text{One sign change:} & s^3 & 1 & -3 \\ & s^2 & 0 \approx \epsilon & 2 \\ & s^1 & -3 - \dfrac{2}{\epsilon} & \\ \text{One sign change:} & s^0 & 2 & \end{array}$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

Summary rules of Routh Test

1. ***A necessary (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial be positive.***
2. ***A system is stable if and only if all the elements in the first column***
3. ***of the Routh array are positive.***

4. *Routh array*

$$\begin{array}{llllll} s^n & : & 1 & a_2 & a_4 & \cdots \\ s^{n-1} & : & a_1 & a_3 & a_5 & \cdots \end{array}$$

We then add subsequent rows to complete the **Routh array**:

$$\begin{array}{lllllll} \text{Row} & n & s^n: & 1 & a_2 & a_4 & \cdots \\ \text{Row} & n-1 & s^{n-1}: & a_1 & a_3 & a_5 & \cdots \\ \text{Row} & n-2 & s^{n-2}: & b_1 & b_2 & b_3 & \cdots \\ \text{Row} & n-3 & s^{n-3}: & c_1 & c_2 & c_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \text{Row} & 2 & s^2: & * & * & & \\ \text{Row} & 1 & s^1: & * & & & \\ \text{Row} & 0 & s^0: & * & & & \end{array}$$

We compute the elements from the $(n-2)$th and $(n-3)$th rows as follows:

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1},$$

$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1},$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1},$$

$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

$$c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1},$$

$$c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}.$$

Application of Routh's Stability Criterion to Control-System Analysis. Routh's stability criterion is of limited usefulness in linear control-system analysis, mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system. It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Consider the system shown in Figure 5–35. Let us determine the range of K for stability. The closed-loop transfer function is

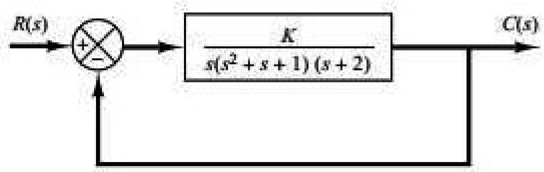
$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{llll} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2 - \frac{9}{7}K & & \\ s^0 & K & & \end{array}$$



For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

Note that the ranges of design parameters that lead to stability may be determined by use of Routh's stability criterion.

4. Self- Test

1. By using Routh Array check the stability of polynomial eqn.
   $a(s) = S^6 + 4S^5 + 3S^4 + 2S^3 + S^2 + 4S + 4$

Another, more rigorous, way to resolve the ambiguity is to use the Nyquist stability criterion, the subject of the next section. However, because the Nyquist criterion is fairly complex, it is important while studying it to bear in mind the theme of this section, namely, that for most systems a simple

relationship exists between closed loop stability and the open-loop frequency response.

3.3 The Nyquist Stability Criterion

The Nyquist stability criterion relates the open-loop frequency response to the number of closed-loop poles of the system in the RHP. Study of the Nyquist criterion will allow you to determine stability from the frequency response of a complex system, perhaps with one or more resonances, where the magnitude curve crosses 1 several times and/or the phase crosses 180◦ several times. It is also very useful in dealing with open-loop unstable systems, no minimum-phase systems, and systems with pure delays (transportation lags).

Consider the transfer function H1（s） whose poles and zeros are indicated in the s-plane in Fig. We wish to evaluate H1 for values of s on the clockwise contour C1. (Hence this is called *contour evaluation*.)

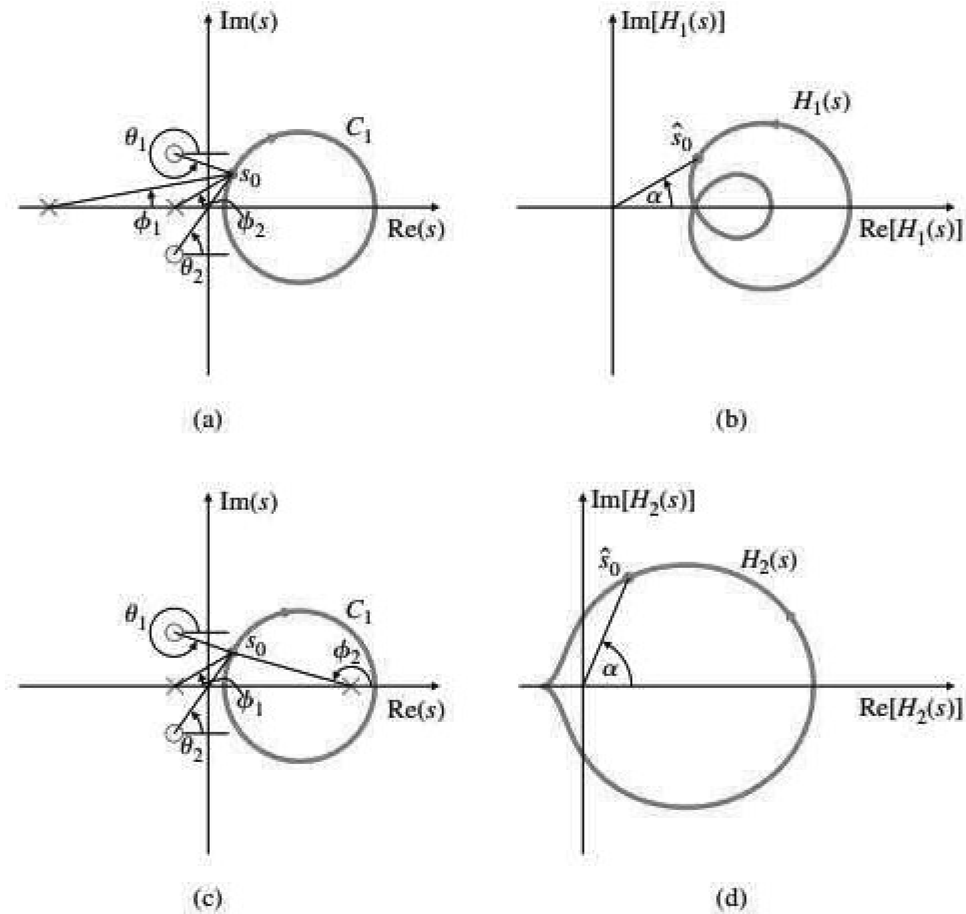


Figure 6.16 Contour evaluations: (a) s-plane plot of poles and zeros of H1(s) and the contour C1; (b) H1(s) for s on C1; (c) s-plane plot of poles and zeros of H2(s) and the contour C1; (d) H2(s) for s on C1

A contour map of a complex function will encircle the origin Z –P times, where Z is the number of zeros and P is the number of poles of the function inside the contour.

$$\alpha = \theta_1 + \theta_2 - (\phi_1 + \phi_2).$$

3.4 Application of The Argument Principle to Control Design

To apply the principle to control design, we let the C_1 contour in the s-plane encircle the entire RHP, the region in the s-plane where a pole would cause an unstable system (Fig. 6.17). The resulting evaluation of $H(s)$ will encircle the origin only if $H(s)$ has an RHP pole or zero.

As stated earlier, what makes all this contour behavior useful is that a contour evaluation of an *open-loop* $KG(s)$ can be used to determine stability of the *closed-loop* system. Specifically, for the system in Fig. 6.18, the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{KG(s)}{1 + KG(s)}.$$

Figure 6.17
An s-plane plot of a contour C_1 that encircles the entire RHP

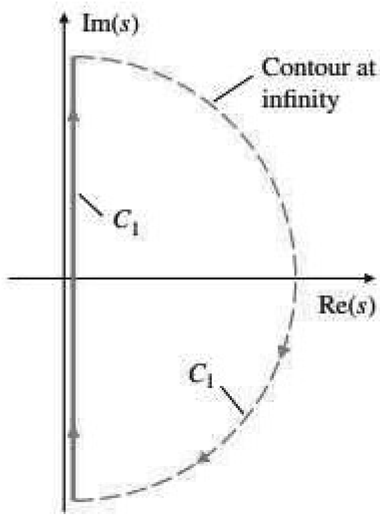
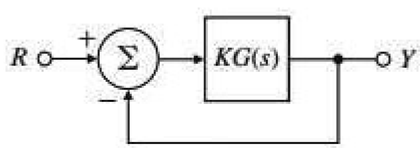


Figure 6.18
Block diagram for $Y(s)/R(s) = KG(s)/[1 + KG(s)]$



Therefore, the closed-loop roots are the solutions of

$1 + KG(s) = 0,$

Procedure for Determining Nyquist Stability

1. Plot $KG(s)$ for $-j\infty \leq s \leq +j\infty$. Do this by first evaluating $KG(j\omega)$ for $\omega = 0$ to ω_h, where ω_h is so large that the magnitude of $KG(j\omega)$ is negligibly small for $\omega > \omega_h$, then reflecting the image about the real axis and adding it to the preceding image. The magnitude of $KG(j\omega)$ will be small at high frequencies for any physical system. The Nyquist plot will always be symmetric with respect to the real axis. The plot is normally created by the NYQUIST Matlab m-file.
2. Evaluate the number of clockwise encirclements of -1, and call that number N. Do this by drawing a straight line in any direction from -1 to ∞. Then count the net number of left-to-right crossings of the straight line by $KG(s)$. If encirclements are in the counterclockwise direction, N is negative.
3. Determine the number of unstable (RHP) poles of $G(s)$, and call that number P.
4. Calculate the number of unstable closed-loop roots Z:

$$Z = N + P. \tag{6.28}$$

For stability we wish to have Z = 0; that is, no characteristic equation roots in the RHP.Let us now examine a rigorous application of the procedure for determining stability using Nyquist plots for some examples.

EXAMPLE 6.8 *Nyquist Plot for a Second-Order System*

Determine the stability properties of the system defined in Fig. 6.20.

Solution. The root locus of the system in Fig. 6.20 is shown in Fig. 6.21. It shows that the system is stable for all values of K. The magnitude of the frequency response of $KG(s)$ is plotted in Fig. 6.22(a) for $K = 1$, and the phase is plotted in Fig. 6.22(b); this is the typical Bode method of presenting frequency response and represents the evaluation of $G(s)$ over the interesting

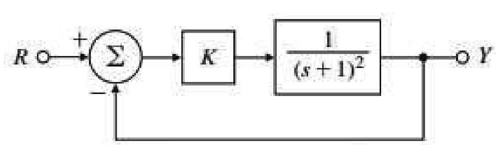


Figure 6.20
Control system for Example 6.8

Figure 6.21
Root locus of $G(s) = \frac{1}{(s+1)^2}$ with respect to K

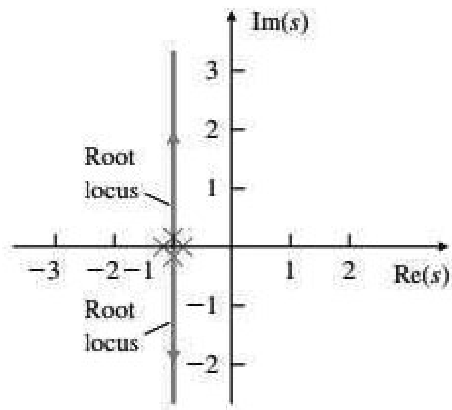
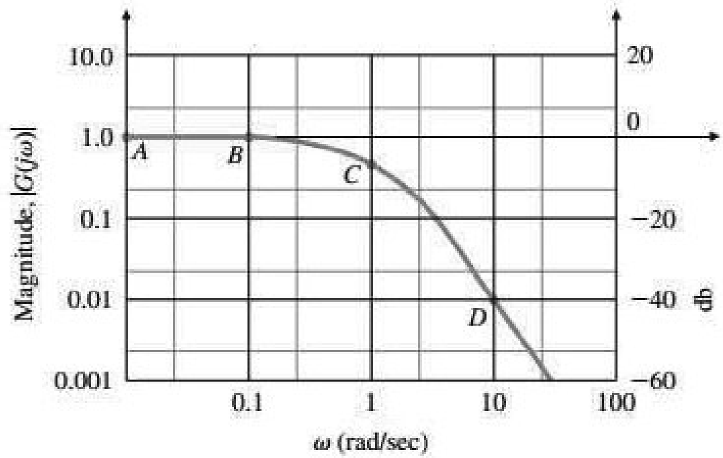
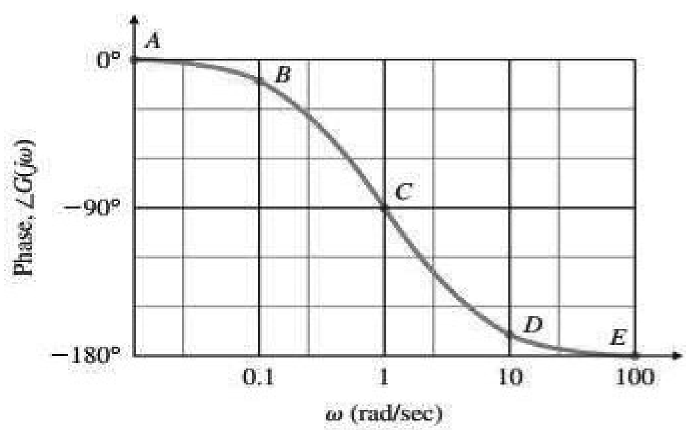


Figure 6.22
Open-loop Bode plot for $G(s) = \frac{1}{(s+1)^2}$



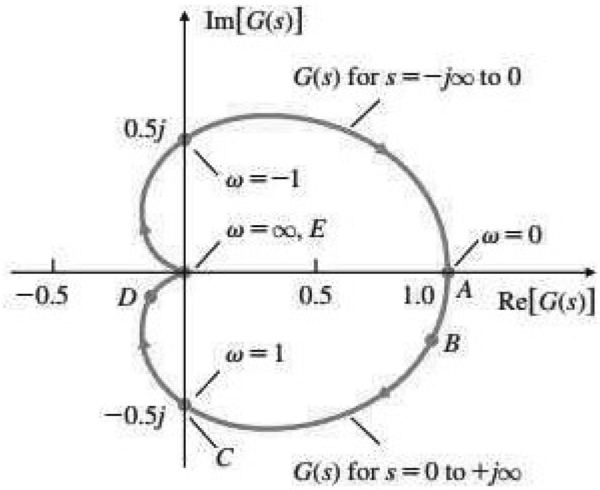
(a)



(b)

range of frequencies. The same information is replotted in Fig. 6.23 in the Nyquist (polar) plot form. Note how the points A, B, C, D, and E are mapped from the Bode plot to the Nyquist plot in Fig. 6.23. The arc from G(s) = +1 ($\omega = 0$) to G(s) = 0 ($\omega = \infty$) that lies below the real axis is derived from Fig. 6.22. The portion of the C1 arc at infinity from Fig. 6.17 transforms into G(s) = 0 in Fig. 6.23; therefore, a continuous evaluation of

Figure 6.23
Nyquist plot[9] of the evaluation of $KG(s)$ for $s = C_1$ and $K = 1$



G(s) with s traversing C1 is completed by simply reflecting the lower arc about the real axis. This creates the portion of the contour above the real axis and completes the Nyquist (polar) plot. Because the plot does not encircle −1, N = 0. Also, there are no poles of G(s) in the RHP, so P = 0. From Eq. (6.28), we conclude that Z = 0, which indicates there are no unstable roots of the closed-loop system for K = 1. Furthermore, different values of K would simply change the magnitude of the polar plot, but no positive value of K would cause the plot to encircle −1, because the polar plot will always cross the negative real axis when KG(s) = 0. Thus the Nyquist stability criterion confirms what the root locus indicated: the closed-loop system is stable for all K > 0.

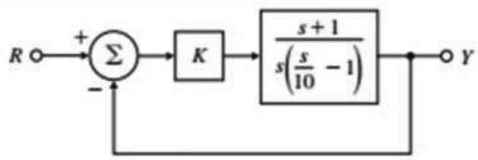
The Matlab statements that will produce this Nyquist plot are

```
s = tf('s');
sysG = 1/(s+1)^2;
nyquist(sysG);
```

5. Post- Test

Determine its stability properties using the Nyquist criterion.

Control system for Example 6.10



Answer Keys
Pre- Test
1. Answer: Option A **2.** Answer: Option A
Self-Test

Routh's Test

The polynomial

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

satisfies the necessary condition for stability since all the $\{a_i\}$ are positive and nonzero. Determine whether any of the roots of the polynomial are in the RHP.

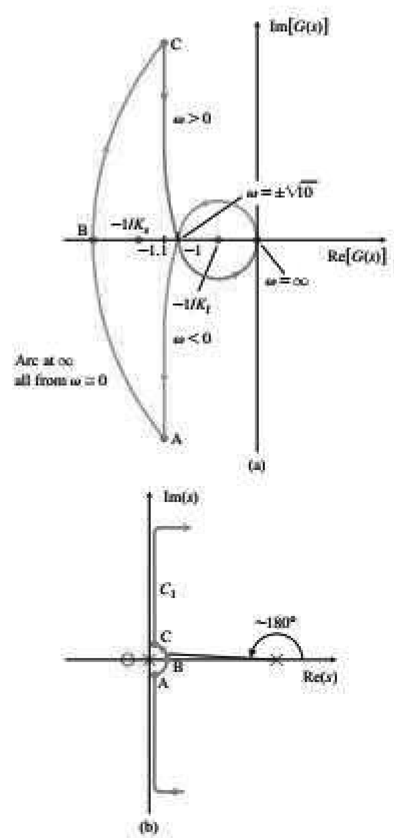
Solution. The Routh array for this polynomial is

s^6:	1	3	1	4
s^5:	4	2	4	0
s^4:	$\frac{5}{2} = \frac{4 \cdot 3 - 1 \cdot 2}{4}$	$0 = \frac{4 \cdot 1 - 4 \cdot 1}{4}$	$4 = \frac{4 \cdot 4 - 1 \cdot 0}{4}$	
s^3:	$2 = \frac{\frac{5}{2} \cdot 2 - 4 \cdot 0}{\frac{5}{2}}$	$-\frac{12}{5} = \frac{\frac{5}{2} \cdot 4 - 4 \cdot 4}{\frac{5}{2}}$	0	

1.

Post- Test

Nyquist plot of $G(s) = \frac{(s+1)}{s(s/10-1)}$, (b) C_1 contour



To draw the Nyquist plot using Matlab, use the following commands:

```
s = tf('s');
sysG = (s + 1)/(s*(s/10 - 1));
nyquist(sysG)
axis([-3 3 -3 3])
```

Ministry of high Education and Scientific Research

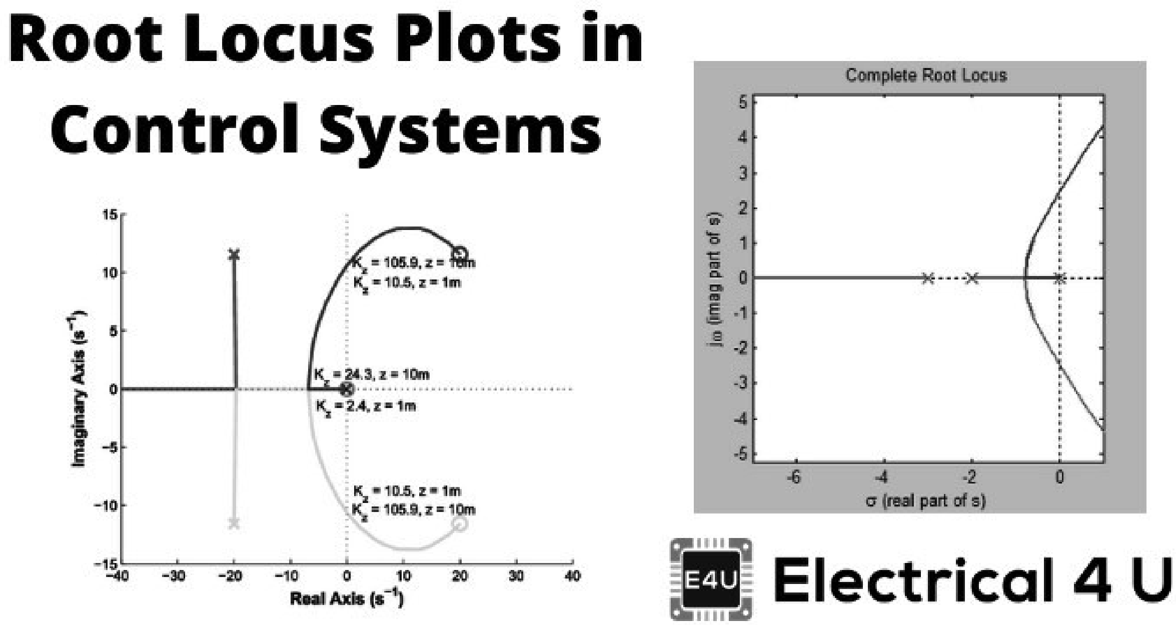
Middle Technical University

Electrical Engineering Technical College

Medical Instrumentation Engineering Techniques Department

- ***Lecture 6 (12th and 13th Weeks)***

ROOT LOCUS TECHNIQUE



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** In studying control systems the reader must be able to model dynamic systems in mathematical terms and analyze their dynamic characteristics.

c. **Central Ideas:** In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

d. **Objectives:** After completing this lecture, the student will be able to:

The main objective of drawing root locus plot is to obtain a clear picture about the transient response of feedback system for various values of open loop gain K and to determine sufficient condition for the value of 'K' that will make the feedback system unstable.

2. Pre-Test:

1. Root locus of s(s+2) +K(s+4) =0 is a circle. What are the coordinates of the center of this circle?

   a) -2,0 b) -3,0 c) -4,0 d) -5,0

2. Which one of the following statements is not correct?

   a) Root loci can be used for analyzing stability and transient performance

   b) Root loci provide insight into system stability and performance

   c) Shape of the root locus gives idea of type of controller needed to meet design specification

   d) Root locus can be used to handle more than one variable at a time

3. Theory:

3.1 Basic concepts of root locus

In the previous sections, we have studies that the stability of a system. It depends on the location of the roots of the characteristic equation. We can also say that the stability of the system depends on the location of closed-loop poles. Such knowledge of the movement of the poles in the s-plane when the parameters are varied is important. The minor changes in the parameters can greatly help in the system designing. The nature of the system's transient response is closely related to the location of the poles in the s-plane. We have also studied the Routh Hurwitz criteria that describe the stability of the algebraic equation. If any of the term in the first column of the Roth table possesses a sign change, the system tends to become unstable.

The root locus method was introduced by W.R Evans in 1948. Root locus is a graphical method in which the movement of poles in the s-plane can be located when a specific parameter is varied from 0 to infinity. The parameter assumed to be varied is generally the gain of the system.

The equation of a closed loop system is given by:

$1 + G(s)H(s) = 0$

Where: G(s) is the gain of the transfer function H(s) is the feedback gain

In the case of root locus, the gain K is also assumed as part of the closed-loop system. K is known as system gain or the gain in the forward path. The characteristic equation after including the forward gain can be represented as:

$1 + KG'(s)H(s) = 0$

Where: $G(s) = KG'(s)$

When the system includes the variable parameter K, the roots of the closed loop system are now dependent on the values of 'K.' The value of 'K' variable can vary in two cases, as shown below:

1. $-\infty$ to $+\infty$
2. 0 to $+\infty$

In the first case, for every different value (integer or decimal) of K, we will get separate set of locations of the roots. If all such locations are joined, the resulting plot is defined as the root locus. We can also define root locus as the locus of the closed loop poles obtained when the system gain 'K' is varied from -infinity to infinity.

When the K varies from zero to infinity, the plot is called the direct root locus. If the system gain 'K' varies from -infinity to zero, the plot thus obtained is known as inverse root locus. The gain K is generally assumed from zero to infinity unless specially stated.

3.2 Root Locus Construction Rules

1.Starting points (K = 0). The root loci start at the open-loop poles.
2. Termination points (K = ∞). The root loci terminate at the open-loop zeros when they exist, otherwise at ∞
3. Number of distinct root loci (branches): This is equal to the order of the characteristic equations (or the number of poles of open loop transfer function).
4. Symmetry of root loci: The root loci are symmetric about the real axis.
5. Root locus locations on the real axis: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
6. Break away (in) points. The points at which a locus breaks away from (or break in) the real axis can be found by letting K as a function of s , taking the derivative of (dK /ds) and then setting the derivative equal to zero.
7. RHS, crossover: This can be obtained by determining the value of K for marginal stability Routh-Hurwitz criterion.

Rules for Constructing a Root Locus

Rule 1

The root locus is symmetric with respect to the real axis.

Rule 2

The root locus originates at the poles of $G(s)H(s)$ (for $K=0$) and terminates at the zeros of $G(s)H(s)$ (as $K \to \infty$), including zeros at infinity.

Rule 3

If the open-loop function has α zeros at infinity, the root locus approaches α asymptotes as $K \to \infty$. The asymptotes are located at angles

$$\theta = \frac{r180^\circ}{\alpha}, \quad r = \pm 1, \pm 3, \pm 5, \ldots$$

and intersect the real axis at the point

$$\sigma_a = \frac{\sum(\text{poles}) - \sum(\text{finite zeros})}{\alpha}. \qquad (\alpha \geq 2 \text{ only})$$

Here, $\alpha = n - m$ (zeros at infinity)

where n = number of poles
m = number of finite zeros

Rule 4

The root locus includes all points on the real axis to the left of an odd number of poles and/or finite zeros.

Rule 5

Breakaway points are given by the roots of

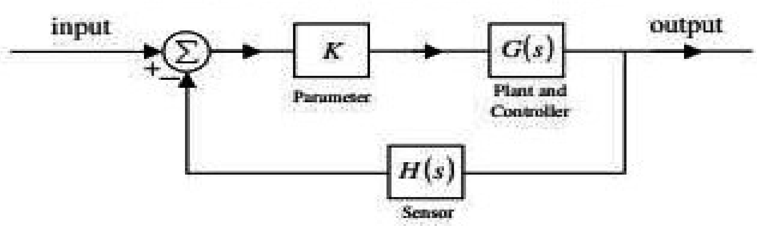
$$\frac{d[G(s)H(s)]}{ds}=0$$

or, equivalently,

$$N(s)D'(s)-D(s)N'(s)=0.$$

Here, $N(s)$ and $D(s)$ are the numerator and denominator polynomials of $G(s)H(s)$.

General Procedure



1. Obtain the open-loop function $KG(s)H(s)$.

2. Quantify the number of poles and finite zeros (n and m) of $G(s)H(s)$ and their locations. Plot the poles and finite zeros in the complex s-plane. (Denote poles and zeros by the symbols $\times$ and o, respectively.)

3. Quantify the number of zeros at infinity (same as the number of asymptotes) using $\alpha=n-m$.

4. Per rule 4, the root locus includes all points on the real axis to the left of an odd number of poles and finite zeros. Include these points in the root locus.

5. If $\alpha\geq 2$, quantify the angles of the α asymptote(s) using

$$\theta=\frac{r180^\circ}{\alpha}, \qquad r=\pm1,\pm3,\pm5,\ldots$$

   Find the point (if it exists) at which the asymptote(s) intersect(s) the real axis using

$$\sigma_a=\frac{\sum(\text{poles})-\sum(\text{finite zeros})}{\alpha}.$$

   Sketch the asymptote(s).

6. Quantify, as appropriate, the breakaway points by calculating the roots of

$$\frac{d[G(s)H(s)]}{ds}=0$$

   or $$N(s)D'(s)-D(s)N'(s)=0.$$

7. Finish sketching the root locus.

Examples:

Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

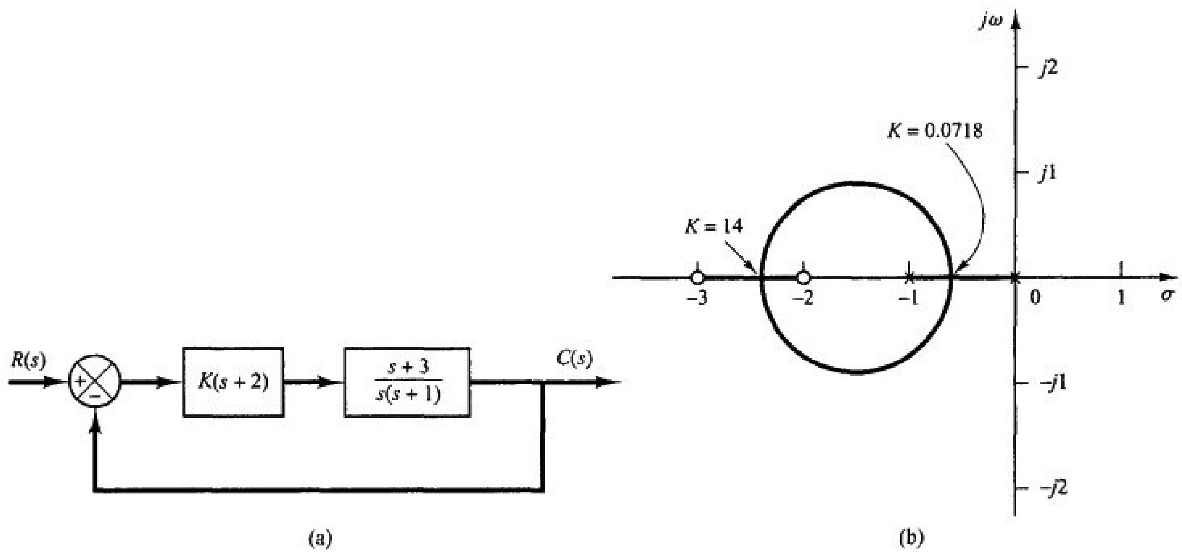


Figure 6–39
(a) Control system; (b) root-locus plot.

3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s + 2)(s + 3)}{s(s + 1)} = 0$$

or

$$K = -\frac{s(s + 1)}{(s + 2)(s + 3)}$$

The breakaway and break-in points are determined from

$$\begin{aligned}\frac{dK}{ds} &= -\frac{(2s + 1)(s + 2)(s + 3) - s(s + 1)(2s + 5)}{[(s + 2)(s + 3)]^2} \\ &= -\frac{4(s + 0.634)(s + 2.366)}{[(s + 2)(s + 3)]^2} \\ &= 0\end{aligned}$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at -1.5 that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–39(b).

Note that this system is stable for any positive value of K since all the root loci lie in the left-half s plane.

Small values of K ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of K ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of K ($14 < K$) correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K.

The value of K should be adjusted so that system performance is optimum according to a given performance index.

Example:

Sketch the root loci of the control system shown in Figure 6–40(a).

Solution. The open-loop poles are located at $s = 0$, $s = -3 + j4$, and $s = -3 - j4$. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

Referring to Equation (6–13), the intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{0 + 3 + 3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

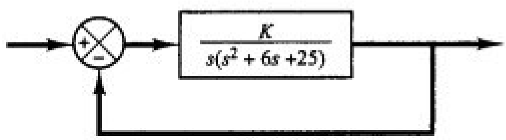
$$K = -s(s^2 + 6s + 25)$$

Now we set

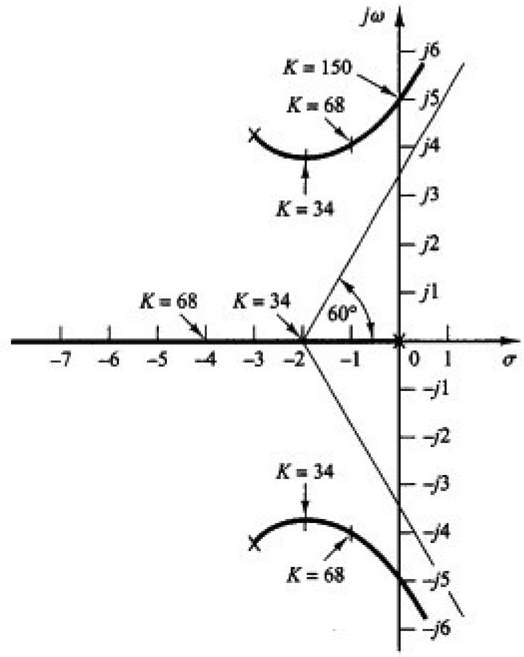
$$\frac{dK}{ds} = -(3s^2 + 12s + 25) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$



(a)



(b)

Figure 6–40
(a) Control system; (b) root-locus plot.

Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of K, we obtain

$$K = -s(s^2 + 6s + 25)\Big|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of K must be real and positive.)

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for ω and K as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \quad K = 150 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain K at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain K is increased, then other closed-loop poles must move toward the left as gain K is increased. This fact can be seen clearly in this problem. If the gain K is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.

4. Self- Test

1. Draw the root locus of the control system having open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

4.1 Effects of Adding Open Loop Poles and Zeros on Root Locus

The root locus can be shifted in ‘s’ plane by adding the open loop poles and the open loop zeros.If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of ‘s’ plane. Because of this, the damping ratio δ decreases. Which implies, damped frequency ωd increases and the time domain specifications like delay time td, rise time tr and peak time tp decrease. But, it effects the system stability.

If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of ‘s’ plane. So, it will increase the control system stability. In this case, the damping ratio δ increases. Which implies, damped frequency ωd decreases and the time domain specifications like delay time td, rise time tr and peak time tp increase.
So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

4.2 Uses of Root Locus

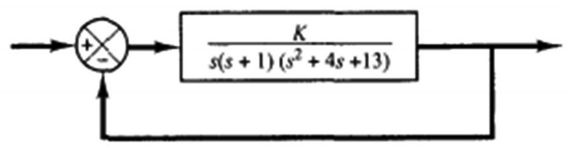
In addition, in determining the stability of the system, root locus also helps to determine:

Damping ratio
The damping ratio is a dimensionless unit that describes how the system decay affects the oscillations of the system.
Natural frequency
It is represented by wn. The value of the system gain K at the location of poles helps in computing the natural frequency and the damping ratio of the system. P, PI, and PID controllers P (proportional), PI (Proportional Integral), and PID (Proportional Integral Derivative) controllers can be designed with the help of root locus technique. Here, the input of the system to be controlled is made proportional to the system gain K.
Lag and lead compensators
The compensators are the additional components in the system added to compensate for deficient performance. The phase lead compensator helps to shift the root locus towards the left in the complex s-plane, and it further increases the system's stability. Similarly, lag and lead compensators can be designed in various ways with the help of the root locus.
Advantages of Root locus
The advantages of root locus are as follows:

We can analyze the absolute stability of the system with the help of a root locus plot.
Using the magnitude and angle conditions, we can find the limiting value of the system gain K for any point on the root locus.
Enhances system designing with better accuracy.
It helps in analyzing the stability of the system with time delay.
Root locus plots help us determine the gain margin, relative stability, phase margin, and the system's settling time.
The root locus technique is easy to implement as compared to other techniques in the control system.
It helps in analyzing the performance of the control system.

5. Post- Test

1. Sketch the root loci for the system shown in Figure 6-42(a).



(a)

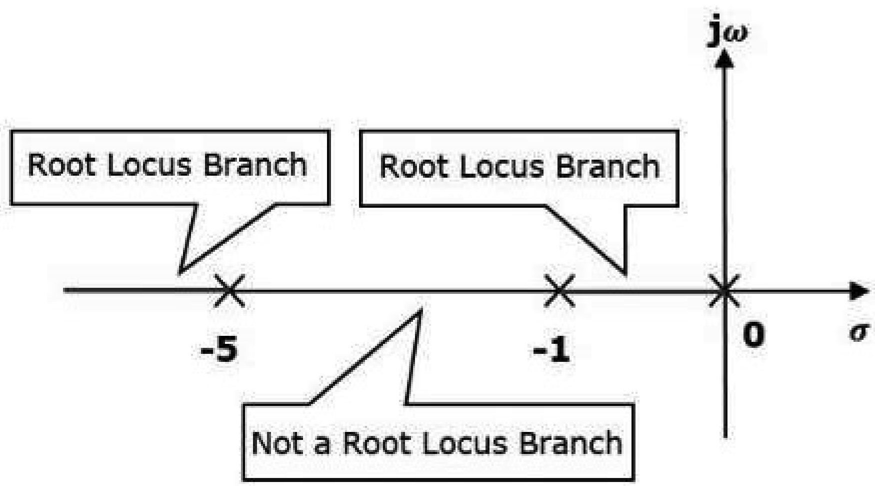
Figure 6–42
(a) Control system;

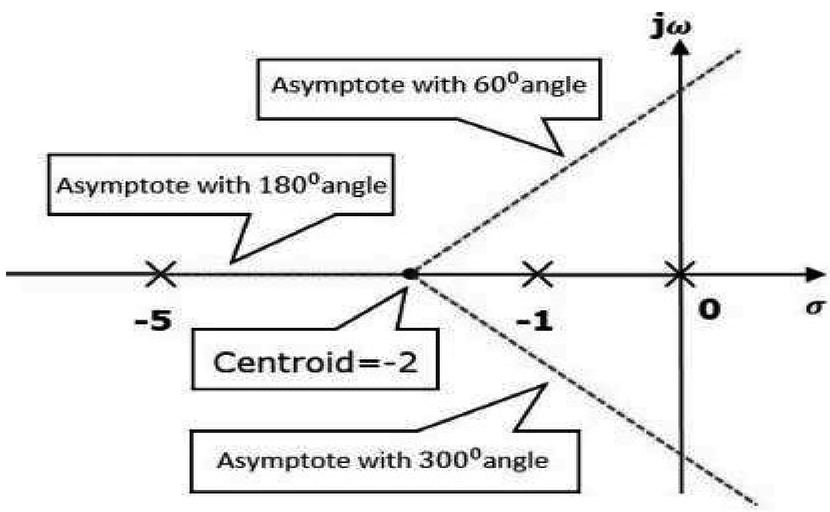
Answer Keys

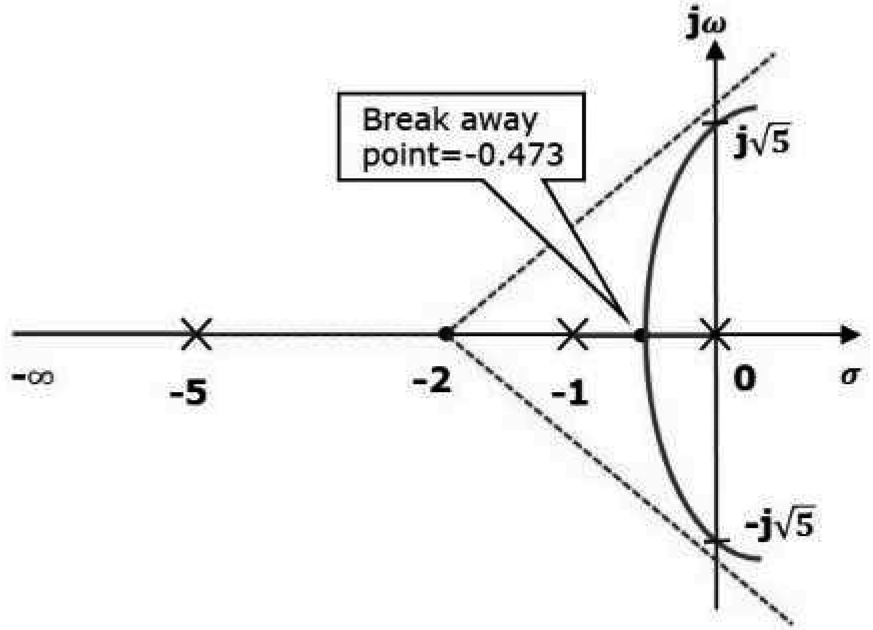
Pre- Test

1. Answer: c
2. Answer: d

Self-Test







Post- Test

Solution. The open-loop poles are located at $s = 0, s = -1, s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k + 1)}{4} = 45^\circ, -45^\circ, 135^\circ, -135^\circ$$

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 1 + 2 + 2}{4} = -1.25$$

The breakaway and break-in points are found from $dK/ds = 0$. Noting that

$$K = -s(s + 1)(s^2 + 4s + 13) = -(s^4 + 5s^3 + 17s^2 + 13s)$$

we have

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 34s + 13) = 0$$

from which we get

$$s = -0.467, \qquad s = -1.642 + j2.067, \qquad s = -1.642 - j2.067$$

Point $s = -0.467$ is on a root locus. Therefore, it is an actual breakaway point. The gain values K corresponding to points $s = -1.642 \pm j2.067$ are complex quantities. Since the gain values are not real positive, these points are neither breakaway nor break-in points.

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180^\circ - 123.69^\circ - 108.44^\circ - 90^\circ$$

or

$$\theta = -142.13^\circ$$

Next we shall find the points where root loci may cross the $j\omega$ axis. Since the characteristic equation is

$$s^4 + 5s^3 + 17s^2 + 13s + K = 0$$

by substituting $s = j\omega$ into it we obtain

$$(j\omega)^4 + 5(j\omega)^3 + 17(j\omega)^2 + 13(j\omega) + K = 0$$

or

$$\left(K + \omega^4 - 17\omega^2\right) + j\omega\left(13 - 5\omega^2\right) = 0$$

from which we obtain

$$\omega = \pm\ 1.6125, \quad K = 37.44 \quad \text{or} \quad \omega = 0, \quad K = 0$$

The root-locus branches that extend to the right-half s plane cross the imaginary axis at $\omega = \pm 1.6125$. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–42(b) shows a sketch of the root loci for the system. Notice that each root-locus branch that extends to the right half s plane crosses its own asymptote.

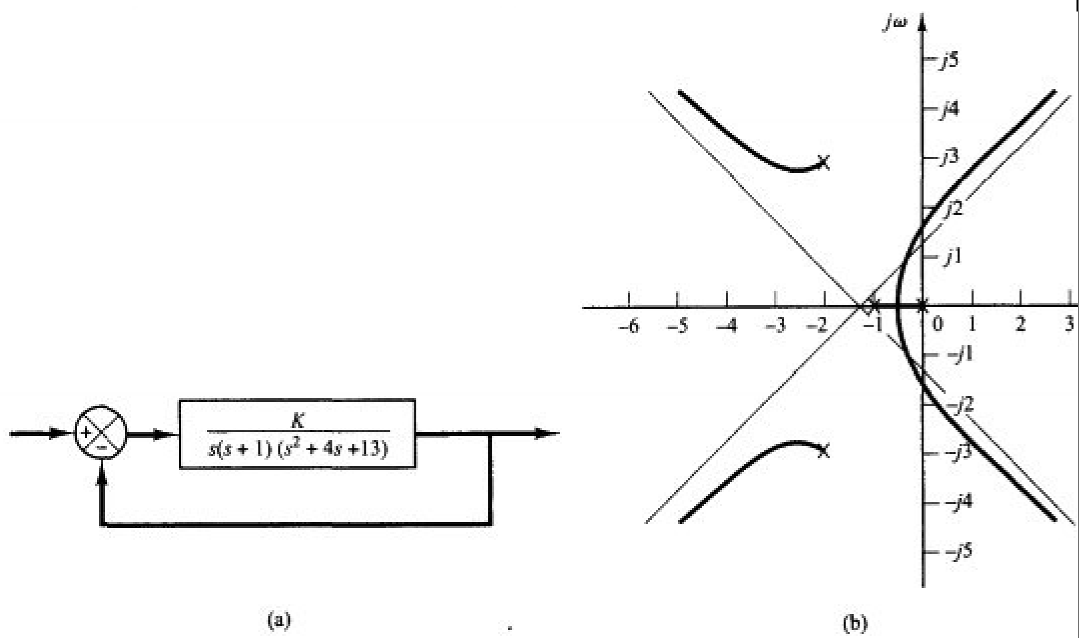


Figure 6–42
(a) Control system; (b) root-locus plot.

Ministry of high Education and Scientific Research

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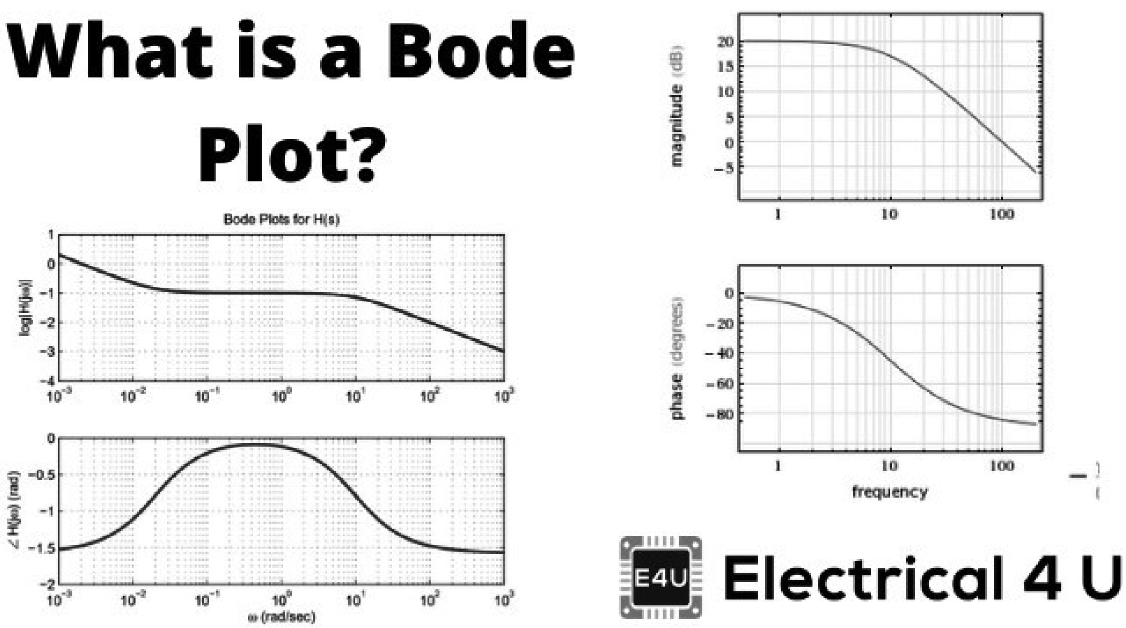
Medical Instrumentation Engineering Techniques Department

- ***Lecture 7+8+9+10 (14th till 24th Week)***

Frequency Domain Analysis, Gain Margin, Phase Margin and Bode Plot



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. Target Population: For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. Rationale: A Control Systems Engineer is responsible for designing, developing, and implementing solutions that control dynamic systems.

c. Central Ideas: Control Systems Engineering is the engineering approach taken to understand how the process can be managed by automation devices and to implement such into operation.

d. Objectives: After completing this lecture, the student will be able to:

1. List the control stability criteria for open loop frequency response.
2. Identify the gain and phase margins necessary for a stable control system.
3. Use a Bode plot to determine if a control system is stable or unstable.
4. Generate Bode plots of control systems they include dead-time delay and determine system stability
5. Describe typical 2_{nd} order lag models found in control systems.
6. Write mathematical formulas for 2_{nd} order lag process models.
7. Compute the parameters of this process model.

2. Pre-Test:

1. A system has poles at 0.01 Hz, 1 Hz and 80Hz, zeroes at 5Hz, 100Hz and 200Hz. The approximate phase of the system response at 20 Hz

a) -90° b) 0° c) 90° d) -180°

2. In a bode magnitude plot, which one of the following slopes would be exhibited at high frequencies by a 4th order all-pole system?

a) -80dB/decade b) -40 dB/decade c) 40 dB/decade d) 80 dB/decade

3. The critical value of gain for a system is 40 and gain margin is 6dB. The system is operating at a gain of:

a) 20 b) 40 c) 80 d) 120

Note: Check your answers in “Answer Keys” in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

3.Theory:

3.1 Bode Plot Stability Criteria

Stable Control System	Open loop gain of less than 1 (G<1 or G<0dB) at open loop phase angle of -180 degrees
Oscillatory Control System Marginally Stable	Open loop gain of exactly 1 (G=1 or G= 0dB) at open loop phase angle of -180 degrees
Unstable Control System	Open loop gain of greater than 1 (G>1 or G>0dB) at open loop phase angle of -180 degrees

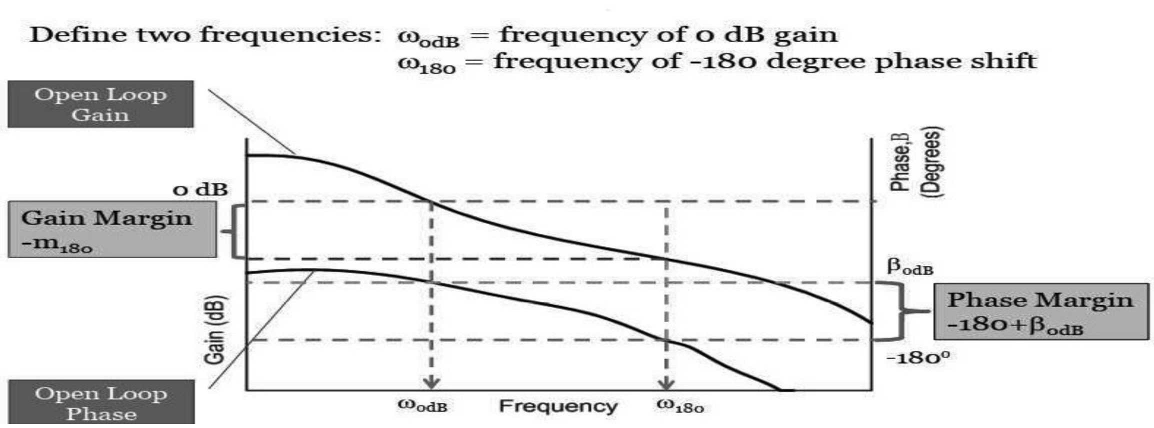
3.2 Phase and Gain Margins

Inherent error and inaccuracies require ranges of phase shift and gain to insure stability.

Gain Margin – Safe level below 1 required for stability Minimum level : G=0.5 or -6 dB at phase shift of 180 degrees

Phase Margin – Safe level above -180 degrees required for stability Minimum level : ϕ=40 degree or -180+ 40=-140 degrees at gain level of 0.5 or 0 dB.

3.3 Determining Phase and Gain Margins

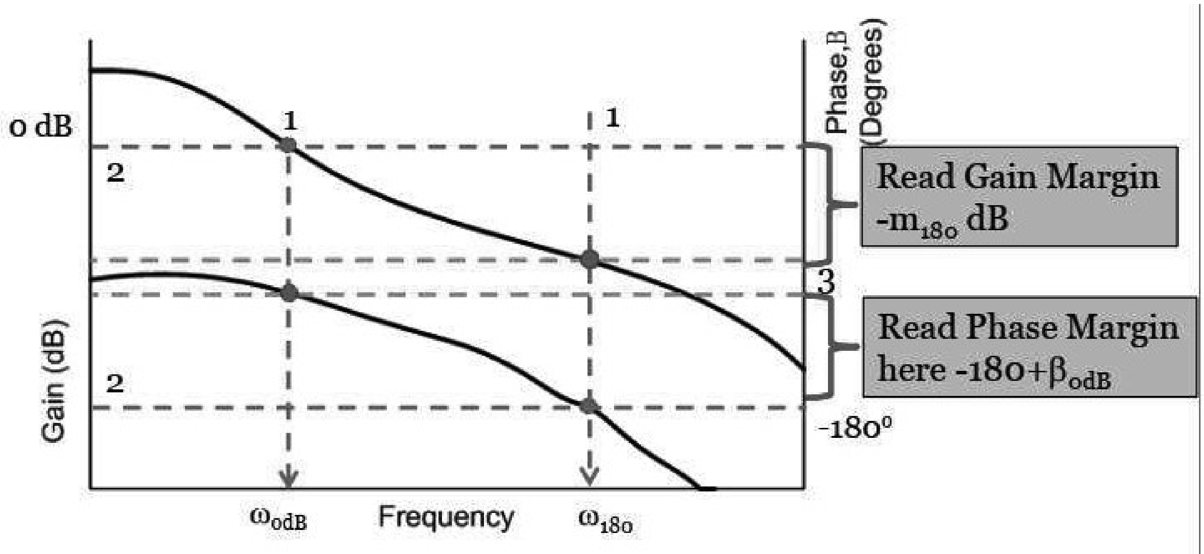


Procedure:

1) Draw vertical lines through 0 dB on gain and -180 on phase plots.

2) Draw horizontal lines through 0 dB and -180 so that they intersect with the vertical lines.

3.) Draw two more horizontal lines that intersect the -180 line on the gain plot and the 0 dB line on the phase plot.



3.4Stability Analysis Using Bode Plots

Bode plot stability analysis is idea for systems with dead-time delay. Delay represented by phase shift that increases with frequency.

Example: A first order lag process has a dead-time delay of 2 seconds and is controlled by a proportional controller. The open loop transfer function is:

$$GH(s) = 40 \cdot \left[\frac{1}{1+100s} \right] \cdot e^{-2s}$$

1)Find the magnitude and phase angle of the transfer function at the following frequencies: w=0.001, 0.01, 0.1 and 1 radian/sec using hand calculations.

2) Use Matlab and construct the Bode plots of the system and then determine the gain and phase margin of the system

Solution (1):

Substitute jω=s $\quad GH(j\omega) = 40\cdot\left[\dfrac{1}{1+100j\omega}\right]\cdot e^{-2j\omega}$

Where $\quad e^{-2j\omega} = \begin{cases} G = 1 \text{ for all } \omega \\ \phi = -2\cdot(57.6)\cdot\omega \end{cases}$

For jω=j0.001 $\quad \left[\dfrac{1}{1+100j0.001}\right] = \left[\dfrac{1}{1+j0.1}\right] = \dfrac{1}{1.005\angle 5.71^\circ} = 0.995\angle -5.71^\circ$

$$e^{-2j0.001} = \begin{cases} G = 1 \text{ for all } \omega \\ \phi = -2\cdot(57.6)\cdot 0.001 \end{cases} = 1\angle -0.115^\circ$$

$$|GH(j0.001)| = 40\cdot 0.995\cdot 1 = 39.8$$

$$\text{ang}[GH(j0.001)] = -5.71^\circ + (-0.115^\circ) = -5.825^\circ$$

Solution (2)

For jω=j0.01

$$GH(j0.01) = \left[\frac{1}{1+100j0.01}\right] = \left[\frac{1}{1+j1}\right] = \frac{1}{1.414\angle 45^\circ} = 0.707\angle -45^\circ$$

$$e^{-2j0.01} = \begin{cases} G = 1 \text{ for all } \omega \\ \phi = -2\cdot(57.6)\cdot 0.01 \end{cases} = 1\angle -1.15^\circ$$

$$|GH(j0.01)| = 40\cdot 0.707\cdot 1 = 28.28$$

$$\text{ang}[GH(j0.01)] = -45^\circ + (-1.15^\circ) = -46.15^\circ$$

For jω=j0.1

$$GH(j0.1) = \left[\frac{1}{1+100j0.1}\right] = \left[\frac{1}{1+j10}\right] = \frac{1}{10.05\angle 84.3^\circ} = 0.0995\angle -84.3^\circ$$

Solution (3)

For jω=j0.1 cont.

$$e^{-2j0.1} = \begin{cases} G = 1 \text{ for all } \omega \\ \phi = -2\cdot(57.6)\cdot 0.1 \end{cases} = 1\angle -11.52^{\circ}$$

$$|GH(j0.1)| = 40\cdot 0.0995\cdot 1 = 3.98$$

$$\text{ang}[GH(j0.1)] = -84.3^{\circ} + (-11.52^{\circ}) = -95.82^{\circ}$$

For jω=j1

$$GH(j1.0) = \left[\frac{1}{1+100j1}\right] = \left[\frac{1}{1+j100}\right] = \frac{1}{100\angle 89.4^{\circ}} = 0.01\angle -89.4^{\circ}$$

$$e^{-2j0.1} = \begin{cases} G = 1 \text{ for all } \omega \\ \phi = -2\cdot(57.6)\cdot 1 \end{cases} = 1\angle -115.2^{\circ}$$

$$|GH(j0.1)| = 40\cdot 0.01\cdot 1 = 0.4 \quad \text{ang}[GH(j0.1)] = -89.4^{\circ} + (-115.2^{\circ}) = -204.6^{\circ}$$

Solution (4)

Calculation summary

Convert all magnitudes to decibels

$$GH(j0.001)_{dB} = 20\log(39.8) = 32\text{ dB}$$

$$GH(j0.01)_{dB} = 20\log(28.28) = 29\text{ dB}$$

$$GH(j0.1)_{dB} = 20\log(3.98) = 12\text{ dB}$$

$$GH(j1)_{dB} = 20\log(0.4) = -8.0\text{ dB}$$

Frequency (rad/sec)	GH	GH (dB)
0.001	$39.8\angle -5.83^{\circ}$	$32\angle -5.83^{\circ}$ dB
0.01	$28.28\angle -46.15^{\circ}$	$29\angle -46.15^{\circ}$ dB
0.1	$3.98\angle -95.82^{\circ}$	$12\angle -95.82^{\circ}$ dB
1.0	$0.4\angle -204.6^{\circ}$	$-8\angle -204.6^{\circ}$ dB

Solution (5)

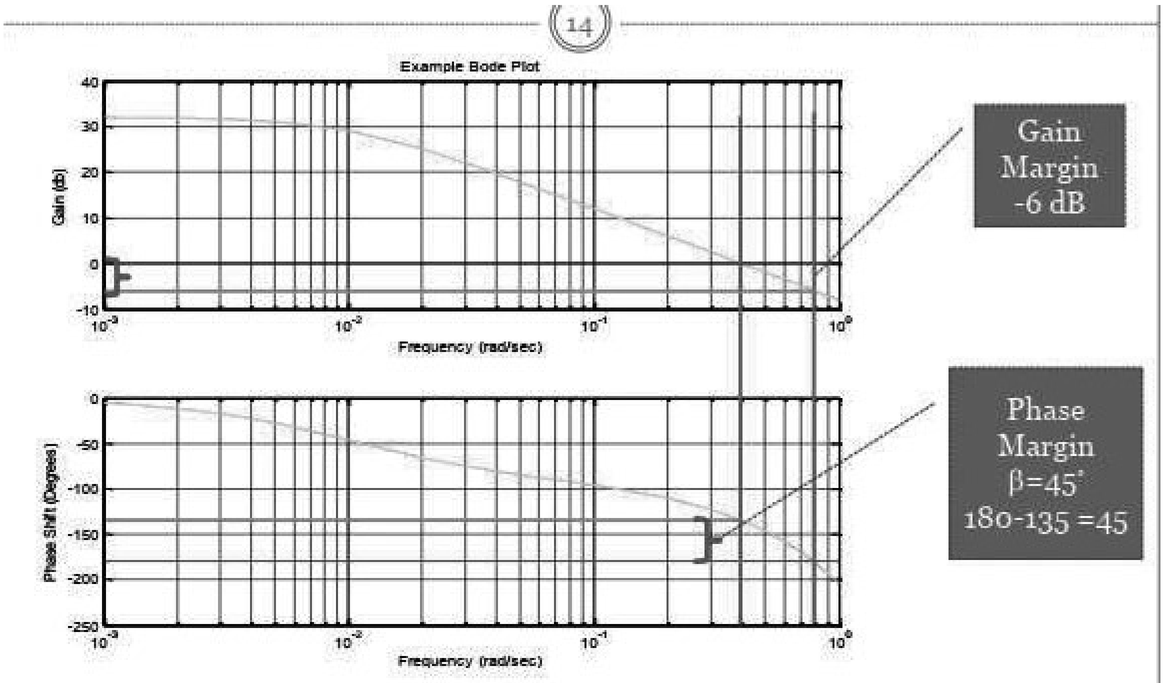
Construct an open-loop Bode plot using MatLAB and find the gain and phase margins for the control system. Example code follows:

```
clear all;
close all;
numgh=[40];                    % define the forward gain numerator and denominator
                               coefficients
demgh=[100 1];
Gh=tf(numgh,demgh);            % construct the transfer function
[m p w]=bode(Gh,{0.001,1}); % Use the bode function with its arguments so that it returns the
                               % magnitude, m, the phase shift, p and the frequencies so that
                               % the effect of the dead time delay can be added to the system
                               % now compute the values of phase shift for the time delay using
                               % the formula -2*w*57.6
pd=-2*w*57.6;
```

Solution (6)

```
% Add the phase shift of the transfer function to the dead-time delay
% take the phase shift out of the 3 column array [m p w]

phase=p(:);
pt=pd+phase;
db=20.*log10(m);  % compute the gain in db

figure;               % create a figure window
subplot(2,1,1);       % divide the plot area in two parts
semilogx(w,db,'go-'); %plot gain in dB on a semilog x-axis
xlabel('Frequency (rad/sec)');  % add labels and title. Turn on the grid.
ylabel('Gain (db)');
title('Example Bode Plot');
grid on;
subplot(2,1,2);  % now do the same for the phase shift plot
semilogx(w,pt,'go-');
xlabel('Frequency (rad/sec)');
ylabel('Phase Shift (Degrees)');
grid on
```

Solution (7)



4. Self- Test

1. The roots of the characteristic equation of the second order system in which real and imaginary part represents the:

a) Damped frequency and damping

b) Damping and damped frequency

c) Natural frequency and damping ratio

d) Damping ratio and natural frequency

2. For relative stability of the system which of the following is sufficient?

A. gain margin B. phase margin C. both a and b D. magnitude

3. Is a part of the human temperature control system?

(A) Digestive system (B) Perspiration system (C) Ear(D) Leg movement

4. Phase margin of a system is used to specify which of the following?

(A) Frequency response (B) Absolute stability (C) Relative stability (D) Time response

5. Which of the following should be done to make an unstable system stable?

(A) The gain of the system should be decreased

(B) The gain of the system should be increased

(C) The number of poles to the loop transfer function should be increased

(D) The number of zeros to the loop transfer function should be increased

3.5 Determining Control Stability Using Bode Plots

Example 21-1: Given the forward gain, G(s), and the feedback system gain, H(s) shown below, find 1) open loop transfer function, 2) closed loop transfer function, 3) error ratio.

$$G(s) = \frac{21.8}{1 + 0.379 \cdot s + 0.0063 \cdot s^2}$$

$$H(s) = \frac{0.356}{1 + 0.478 \cdot s}$$

4) compute the values of the open/closed loop transfer functions when ω=0.1, 1, 10 and 100 rad/sec. 5) compute the value of the error ratio when ω=0.1, 1, 10 and 100 rad/sec. 6) Use MatLAB to plot the open and closed loop transfer function responses on the same axis.

1) Open Loop Transfer Function

$$G(s)H(s) = \left[\frac{21.8}{1+0.379s+0.0063s^2}\right]\left[\frac{0.356}{1+0.478s}\right]$$

Expand the denominator

$$(1+0.379s+0.00635^2)(1+0.478s)$$

$$1+0.379s+0.00635^2+0.478s+0.18125^2+3.01\times10^{-3}s^3$$

$$1+0.857s+0.18746s^2+0.00301s^3$$

$$G(s)H(s) = \boxed{\frac{7.761}{1+0.857s+0.18746s^2+0.00301s^3}}$$

Ans

2) Find closed loop transfer function

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{\dfrac{7.761}{1+0.857s+0.18746s^2+0.00301s^3}}{1+\left[\dfrac{7.761}{1+0.857s+0.18746s^2+0.00301s^3}\right]}$$

Multiply numerator and denominator by $1+0.857s+0.18746s^2+0.00301s^3$ and simplify

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{7.761}{1+0.857s+0.18746s^2+0.00301s^3+7.761}$$

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \boxed{\frac{7.761}{8.761+0.857s+0.18746s^2+0.00301s^3}}$$

Ans

Error ratio calculations

$$ER(s)=\frac{1}{1-\dfrac{7.761^2}{(1+0.867s+0.18746s^2+0.003015s^3)^2}}$$

$$ER(s)=\frac{(1+0.867s+0.18746s^2+0.003015s^3)^2}{(1+0.867s+0.18746s^2+0.003015s^3)^2+7.761^2}$$

$$\boxed{ER(s)=\frac{(1+0.867s+0.18746s^2+0.003015s^3)^2}{(1+0.867s+0.18746s^2+0.003015s^3)^2-60.23}}$$

Ans

4) Compute the values of the open and closed loop transfer functions for ω=0.1 1 10 100 rad/s Substitute jω for s

Open loop

$$G(j\omega)=\frac{21.8}{1+0.379j\omega+0.0063(j\omega)^2}$$

NOTE: $j^2=-1$, $j^3=-j$

$$H(j\omega)=\frac{0.356}{1+0.478j\omega}$$

$$GH(j\omega)=G(j\omega)H(j\omega) \qquad \omega=0.1$$

$$G(j0.1)=\frac{21.8}{1+0.379(j0.1)+0.0063(j0.1)^2}=\frac{21.8}{1+0.0379j-0.000063}=\frac{21.8}{0.999+0.0379j}$$

$$H(j0.1) = \frac{0.356}{1+0.428j0.1} = \frac{0.356}{1+0.0428j} = 0.355\angle{-2.74^\circ}$$

$$G(j0.1) = 21.79 - j0.827 = 21.81\angle{-2.17^\circ}$$

$$G(j0.1)H(j0.1) = (21.81\angle{-2.17^\circ})(0.355\angle{-2.74^\circ})$$

$$G(j0.1)H(j0.1) = \boxed{7.74\angle{-4.91^\circ}}$$

Now for ω=1 rad/sec

$$G(j1) = \frac{21.8}{1+0.379j1+0.0063(j1)^2} = \frac{21.8}{1+0.379j-0.0063} = \frac{21.8}{0.994+0.379j}$$

$$G(j10) = 0.556 - 5.698j = 5.725\angle{-84.43^\circ}$$

$$H(j10) = \frac{0.356}{1+0.428j10} = \frac{0.356}{1+4.28j} = 0.073\angle{-78.1^\circ}$$

$$G(j100) = \frac{21.8}{1+0.379j100+0.0063(j100)^2} = \frac{21.8}{1+37.9j-63} = \frac{21.8}{-62+37.9j}$$

For ω=100 rad/sec

$$G(j100) = \frac{21.8}{1+0.379j100+0.0063(j100)^2} = \frac{21.8}{1+37.9j-63} = \frac{21.8}{-62+37.9j}$$

$$G(j100) = -0.256 - 0.156j = 0.3\angle{-148.64^\circ}$$

$$H(j100) = \frac{0.356}{1 + 0.478j100} = \frac{0.356}{1 + 47.8j} = 1.56\times10^{-4} - 7.44\times10^{-3}j$$

$$H(j100) = 7.446\times10^{-3}\angle{-88.8^\circ}$$

$$G(j100)H(j100) = (0.3\angle{-148.64^\circ})(7.446\times10^{-3}\angle{-88.8^\circ}) = \boxed{0.00223\angle{127.6^\circ}}$$

Convert all gain values into dB

$\omega = 0.1$	$20\text{Log}(7.746) = 17.78\text{dB}$	-4.91° Phase
$\omega = 1$	$20\text{Log}(6.598) = 16.37\text{dB}$	-96.4° Phase
$\omega = 10$	$20\text{Log}(0.418) = -7.59\text{dB}$	-162.5° Phase
$\omega = 100$	$20\text{Log}(0.00223) = -53.0\text{dB}$	127.6° Phase

4) Compute the close loop response using the previously calculated values of G(s)H(s)

Define $GH_c(s) = \frac{G(s)H(s)}{1 + G(s)H(s)}$ and $GH_c(j\omega) = \frac{G(j\omega)H(j\omega)}{1 + G(j\omega)H(j\omega)}$

$j\omega = 0.1j \qquad G(j0.1)H(j0.1) = 7.74\angle{-4.91^\circ}$

$$GH_c(j0.1) = \frac{7.74\angle -4.91^\circ}{1 + 7.74\angle -4.91^\circ}$$

$$GH_c(j0.1) = \boxed{0.886\angle -0.56^\circ}$$

$$j\omega = 1j \qquad G(j1)H(j1) = 6.598\angle -46.48^\circ$$

$$GH_c(j1) = \frac{6.598\angle -46.48^\circ}{1 + 6.598\angle -46.48^\circ}$$

$$GH_c(j1) = \boxed{0.901\angle -5.68^\circ}$$

$$j\omega = 10j \quad G(j10)H(j10) = 0.418\angle{-162.58^\circ}$$

$$GH_c(j10) = \frac{0.418\angle{-162.58^\circ}}{1 + 0.418\angle{-162.58^\circ}}$$

$$GH_c(j10) = \boxed{0.681\angle{-150.79^\circ}}$$

$$j\omega = 100j \quad G(j100)H(j100) = 0.00223\angle{127.6^\circ}$$

$$GH_c(j100) = \frac{0.00223\angle{127.5^\circ}}{1 + 0.00223\angle{127.5^\circ}}$$

$$GH_c(j100) = \boxed{0.00223\angle{127.5^\circ}}$$

Convert all gain values into dB

$$20\log(GH_c(j0.1)) = 20\log(0.886) = -1.051\text{ dB} \quad -0.56^\circ \text{ Phase}$$

$$20\log(GH_c(j1)) = 20\log(0.901) = -0.911\text{ dB} \quad -5.68^\circ \text{ Phase}$$

$$20\log(GH_c(j10)) = 20\log(0.681) = -3.36\text{ dB} \quad -150.7^\circ \text{ Phase}$$

$$20\log(GH_c(j100)) = 20\log(0.00223) = -53\text{ dB} \quad +127.5^\circ \text{ Phase}$$

5) Compute the values of the error ratio

$$ER = \left| \frac{1}{[1+G(s)H(s)][1-G(s)H(s)]} \right|$$

Use open loop values to compute values of ER at given frequencies

At $j\omega = j0.1$ $\quad G(j\omega)H(j\omega) = 7.746\angle{-4.91^\circ}$

$$ER = \left| \frac{1}{(1 + 7.746\angle{-4.91^\circ})(1 - 7.746\angle{-4.91^\circ})} \right|$$

$$ER = \left| \frac{1}{59.015\angle{170^\circ}} \right| = \boxed{0.017}$$

Now for ω=1 rad/sec

$$G(j\omega)H(j\omega) = 6.598\angle{-46.48^\circ}$$

$$ER = \left| \frac{1}{(1 + 6.598\angle{-46.48^\circ})(1 - 6.598\angle{-46.48^\circ})} \right|$$

$$ER = \left| \frac{1}{43.41\angle{85.9}} \right| = \boxed{0.023}$$

For ω=10 rad/sec $\quad G(j\omega)H(j\omega) = 0.00223\angle{127.5^\circ}$

$$ER = \left| \frac{1}{(1 + 0.00223\angle{127.5^\circ})(1 - 0.00223\angle{127.5^\circ})} \right|$$

$$ER = \left| \frac{1}{1\angle{2.62^\circ}} \right| = \boxed{1}$$

Convert all gain values into dB

$j\omega = j0.1 \quad 20\log(0.017) = -35.4\ dB$

$j\omega = j1 \quad 20\log(0.023) = -32.8\ dB$

$j\omega = j10 \quad 20\log(1.159) = 1.28\ dB$

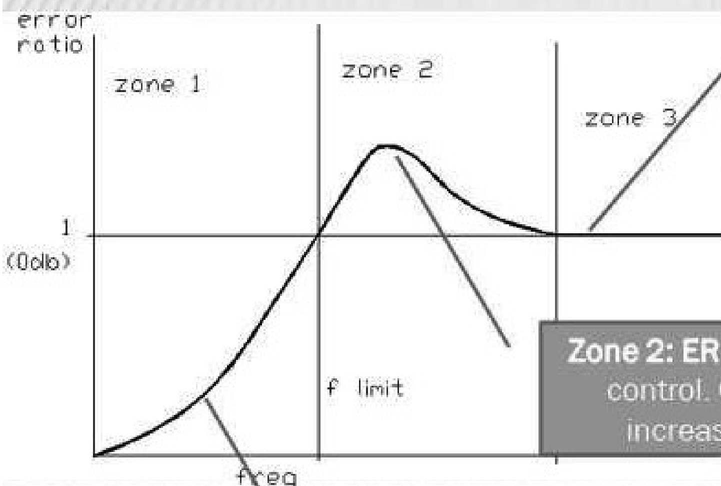
$j\omega = j100 \quad 20\log(1) = 0\ dB$

System becomes uncontrollable between these two frequencies

Error ratio magnitude increases as frequency increases. It peak and becomes a constant value of 1 (0 dB)

INTERPRETING ERROR RATIO PLOTS

Define control zones



Zone 3: ER-=0 dB no control. Controller does not change error

Zone 2: ER> 0 dB poor control. Controller increases error

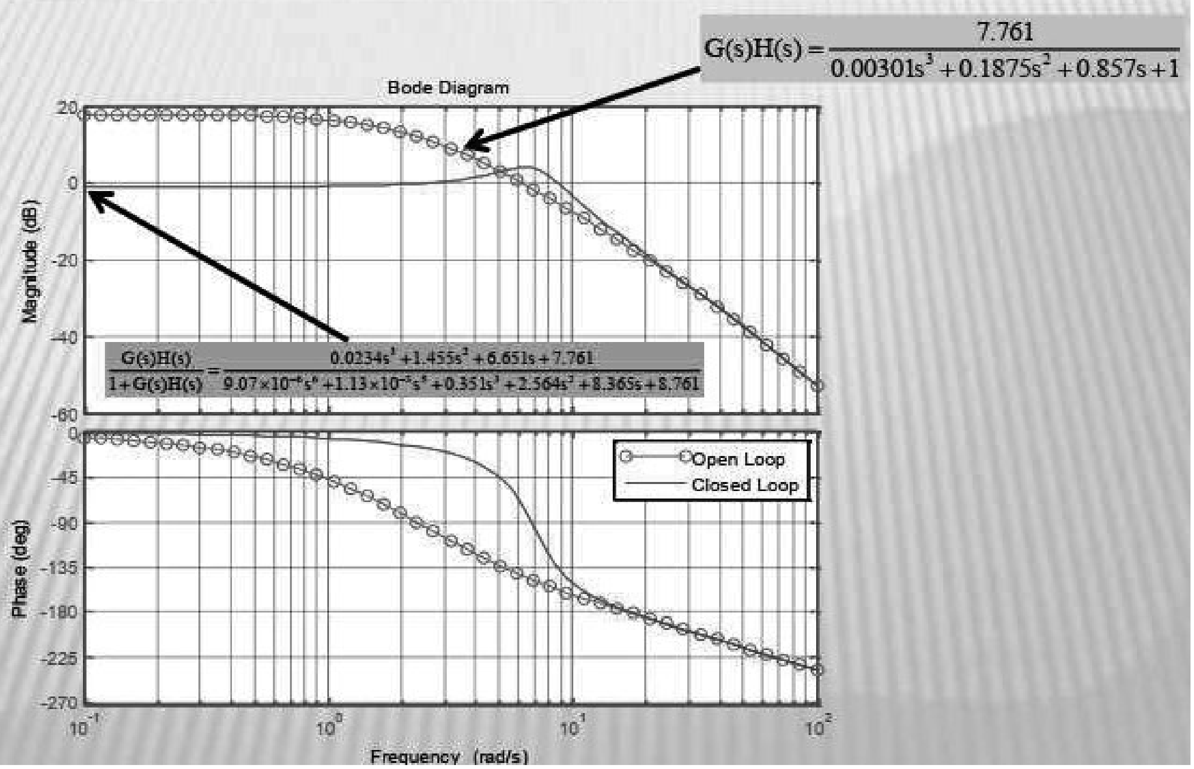
Zone 1: ER< 0 dB good control. Controller

GENERATING PLOTS USING MATLAB

Use MatLAB script to create open and closed loop Bode plots of example system

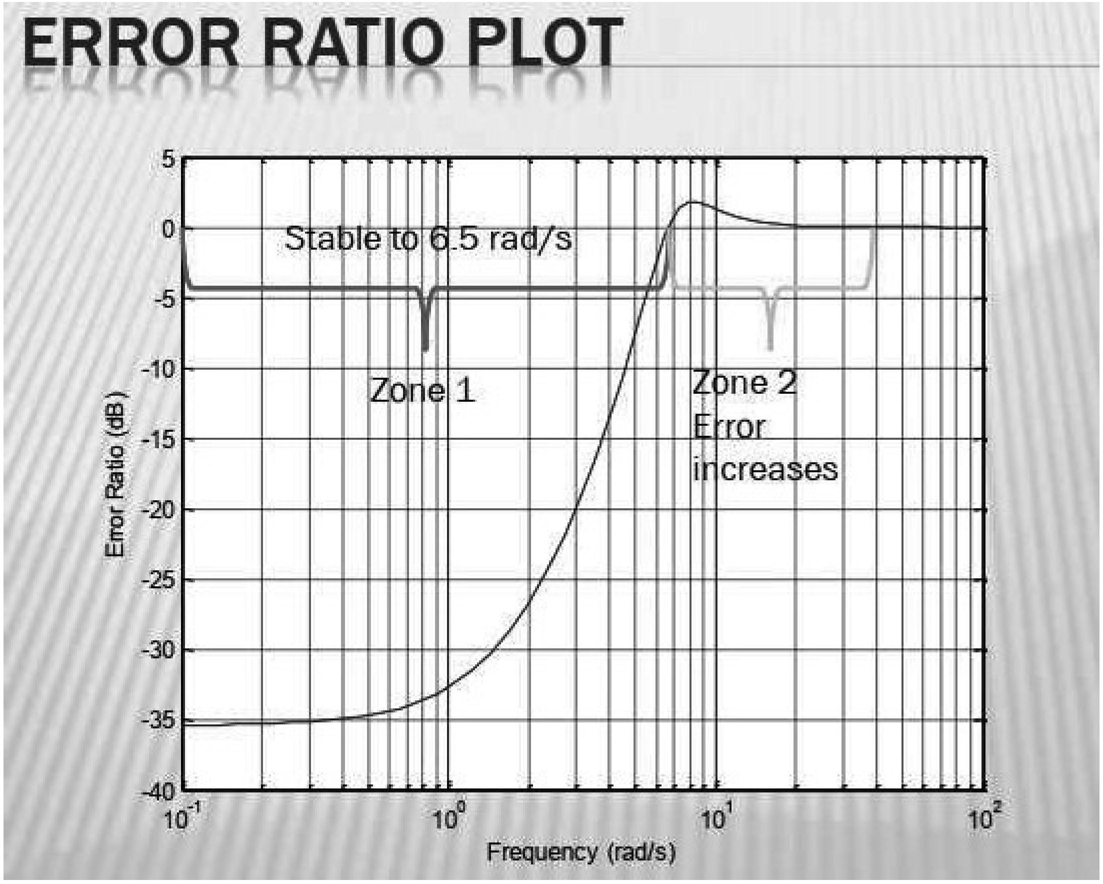
```
% Example bode calculations
clear all;
close all;
% define the forward gain numerator and denominator coefficients
numg=[21.8];
demg=[0.0063 0.379 1];
% define the feedback path gain numerator and denominators
numh=[0.356];
demh=[0.478 1];
% construct the transfer functions
G=tf(numg,demg);
H=tf(numh,demh);
% find GH(s)
GH=G*H
% find the closed loop transfer function
GHc=GH/(1+GH)

% The value in curly brackets are freq. limits
bode(GH,'go-',GHc,'r-',{0.1,100})
```

BODE PLOTS OF EXAMPLE 21-1



MATLAB CODE FOR ERROR PLOT EXAMPLE 21-1

```
% Example Error Ratio calculations
clear all;
close all;
% define the forward gain numerator and denominator coefficients
numg=[21.8];
demg=[0.0063 0.379 1];
% define the feedback path gain numerator and denominators
numh=[0.356];
demh=[0.478 1];
% construct the transfer functions
G=tf(numg,demg);
H=tf(numh,demh);
% find GH(s)
GH=G*H;
% find the error ratio
 ER=1/((1+GH)*(1-GH));
 [mag,phase,W]=bode(ER,{0.1,100}); %Use bode plot with output sent to arrays
N=length(mag); %Find the length of the array
gain=mag(1,1:N); %Extract the magnitude from the mag array
db=20.*log10(gain); % compute the gain in dB and plot on a semilog plot
semilogx(W,db);
grid on; %Turn on the plot grid and label the axis
xlabel('Frequency (rad/s)');
ylabel('Error Ratio (dB)');
```



2.4 Second-Order Lag Processes

Characteristics: Two energy storage elements
System response determined by three parameters: steady-state gain-G, damping ratio ζ, and resonant frequency, ω_o

Examples: 2 capacitances,
1 mass and 1 spring
1 capacitance and 1 inductance

3.6 General Second Order Lag Process Equations

Time domain equation:

$$A_2 \cdot \frac{d^2y}{dt^2} + A_1 \cdot \frac{dy}{dt} + y = G \cdot x$$

Transfer function:

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2}$$

Parameters in terms of coefficients A_1 and A_2

Coefficients in terms of Parameters ζ and ω_0

$$\omega_0 = \sqrt{\frac{1}{A_2}} \qquad \zeta = \frac{A_1}{2 \cdot \sqrt{A_2}} = \frac{A_1 \cdot \omega_0}{2} \qquad A_2 = \frac{1}{\omega_0^{\,2}} \qquad A_1 = \frac{2 \cdot \zeta}{\omega_0}$$

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2} \qquad A_2 = \frac{1}{\omega_0^{\,2}} \qquad A_1 = \frac{2 \cdot \zeta}{\omega_0}$$

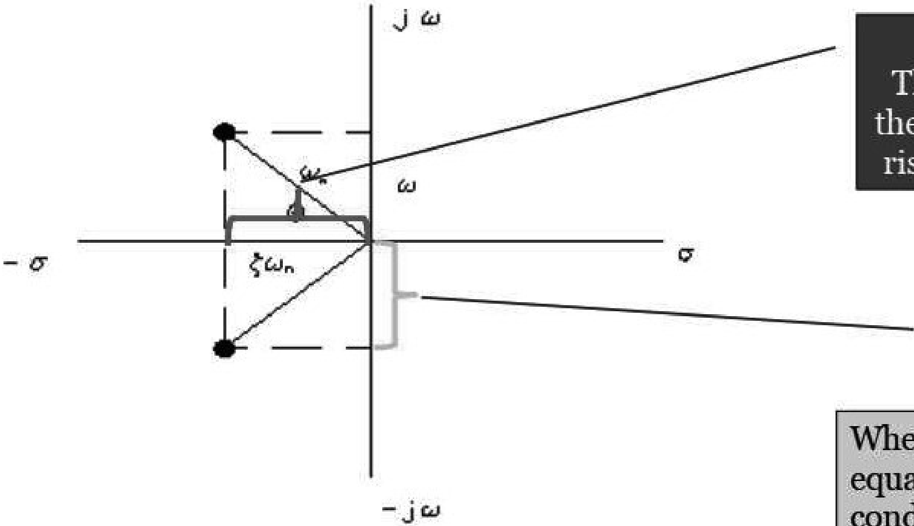
Combine these two equation and simplify

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + \left[\frac{2 \cdot \zeta}{\omega_0}\right] \cdot s + \left[\frac{1}{\omega_0^{\,2}}\right] \cdot s^2} \qquad \frac{Y(s)}{X(s)} = \left[\frac{\omega_0^{\,2}}{\omega_0^{\,2}}\right] \left[\frac{G}{1 + \left[\frac{2 \cdot \zeta}{\omega_0}\right] \cdot s + \left[\frac{1}{\omega_0^{\,2}}\right] \cdot s^2}\right]$$

$$\frac{Y(s)}{X(s)} = \frac{G \cdot \omega_0^{\,2}}{\omega_0^{\,2} + 2 \cdot \zeta \cdot \omega_0 \cdot s + s^2}$$

Characteristic Equation: Roots determine system response

3.7 Second Order System Responses



$\omega_o \zeta = \sigma$
This controls the exponential rise and decay

$\omega_0 \cdot \sqrt{\zeta^2 - 1}$

When 0<ζ<1 above equation determines conditional frequency- a damped sinusoid

Find roots to characteristic equation

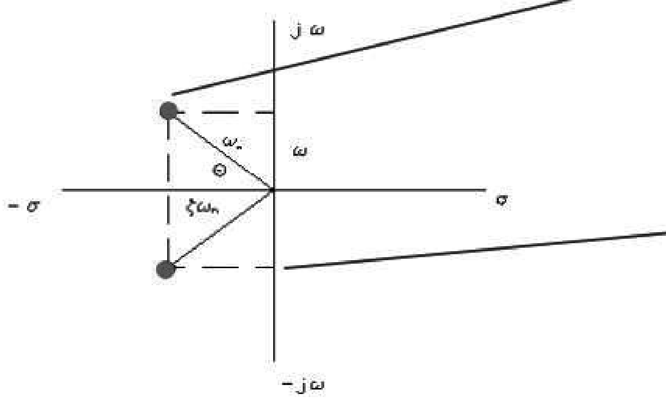
$$\omega_0^{\ 2} + 2 \cdot \zeta \cdot \omega_0 \cdot s + s^2 = 0$$

$$s_1 = -\zeta \cdot \omega_0 + j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \sigma_1 + j \cdot \omega_1$$

$$s_2 = -\zeta \cdot \omega_0 - j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \sigma_2 + j \cdot \omega_2$$

Two poles at these locations

Plot these roots on complex plane

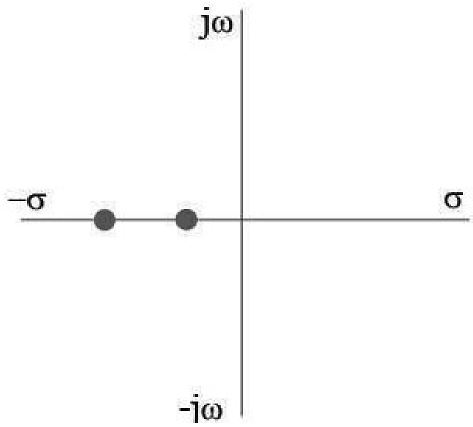


As poles near imaginary axis system become more oscillatory

If ζ=0, damping is zero and system will oscillate at $\omega = \omega_o$

Roots of quadratic formula can have three possible forms

1) real - distinct
2) real - identical
3) imaginary - conjugate pairs



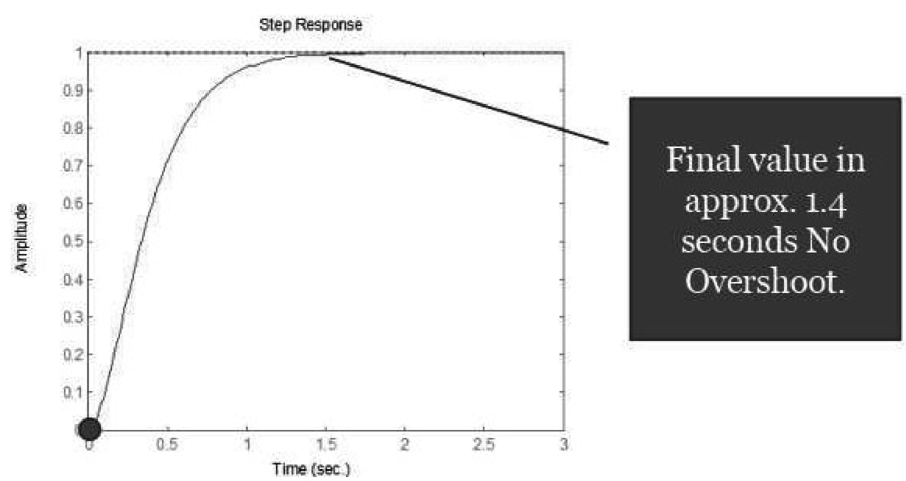
Location of roots is controlled by the values of ζ and ω_o.

If natural frequency is constant them damping controls system response

Damping coefficient value and system responses

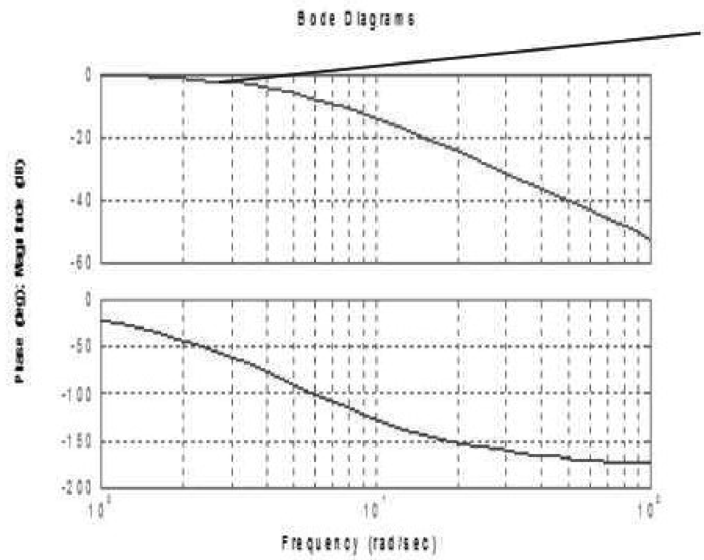
ζ = 1 - critically damped system. Reaches the final value the fastest without having any overshoot. Roots are equal and real.

Time response



Second Order System Responses-Critically Damped

Bode plot of Critically damped system

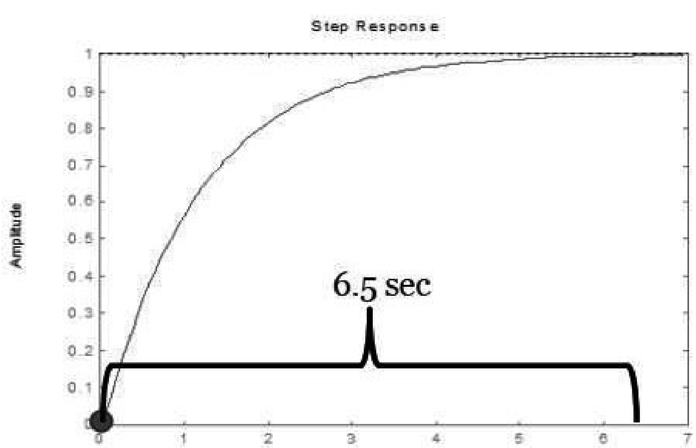


Two poles at this point -3 dB from max. gain

Response similar to lag process

Second Order Response-Over Damped System

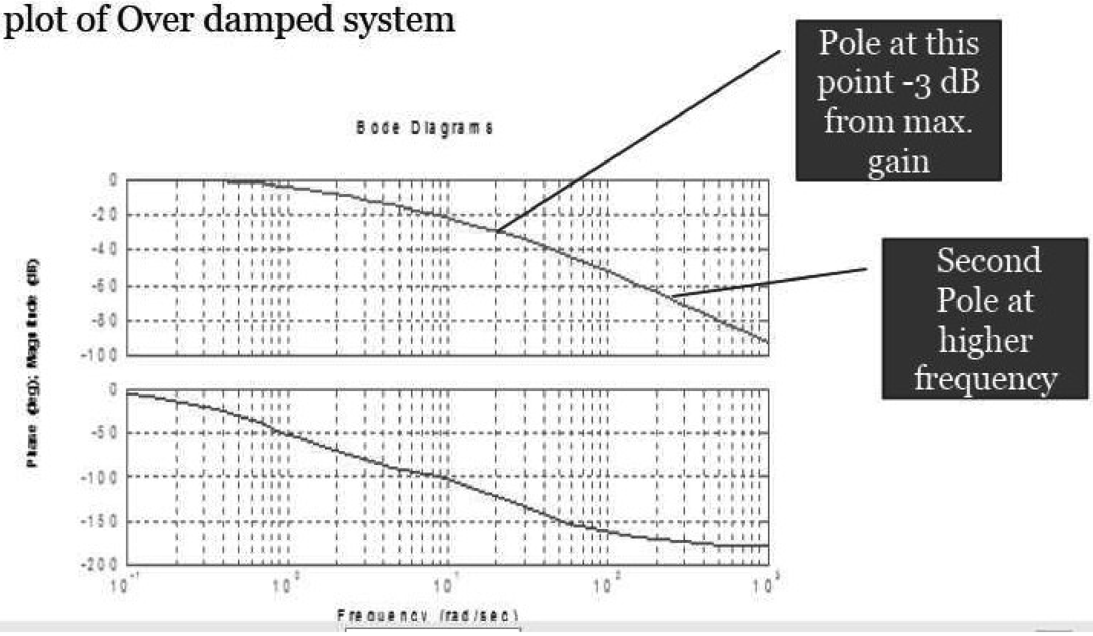
$\zeta > 1$ - over damped system. Reaches the final value slowly but with no overshoot. More damping, slower response to final value. Roots are real but not equal.



Compared to the critically damped case, the response time is slower. Approx. 6.5 sec to get to final value vs 1.4 sec

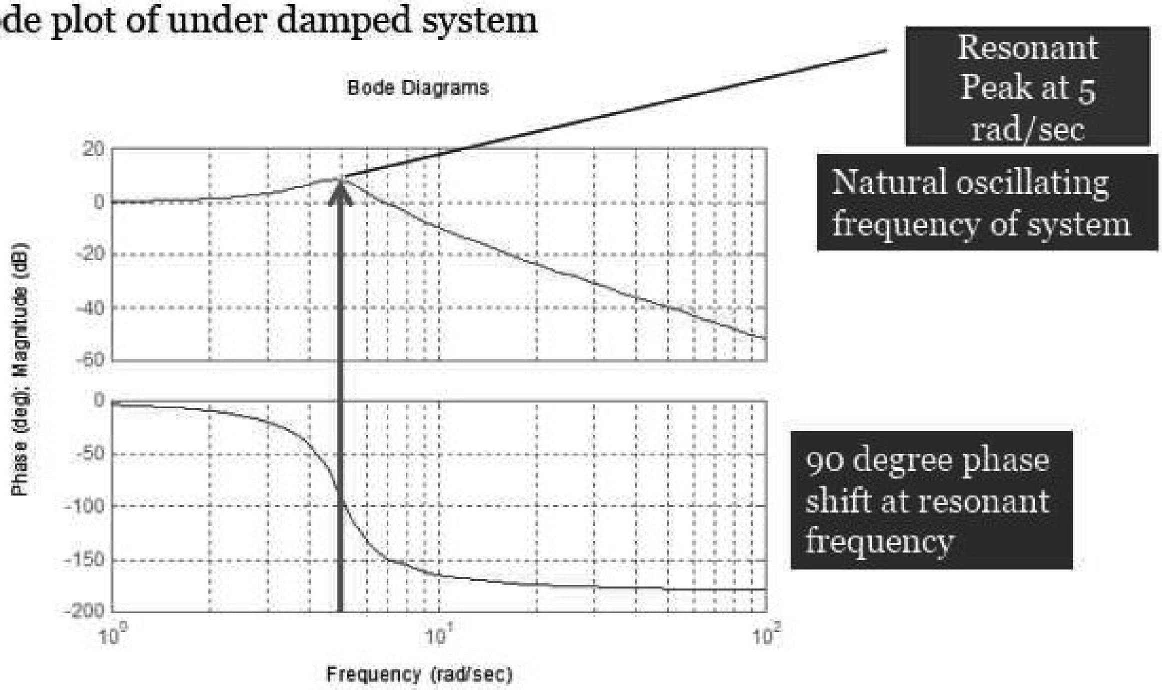
Over Damped System Frequency Response

Bode plot of Over damped system



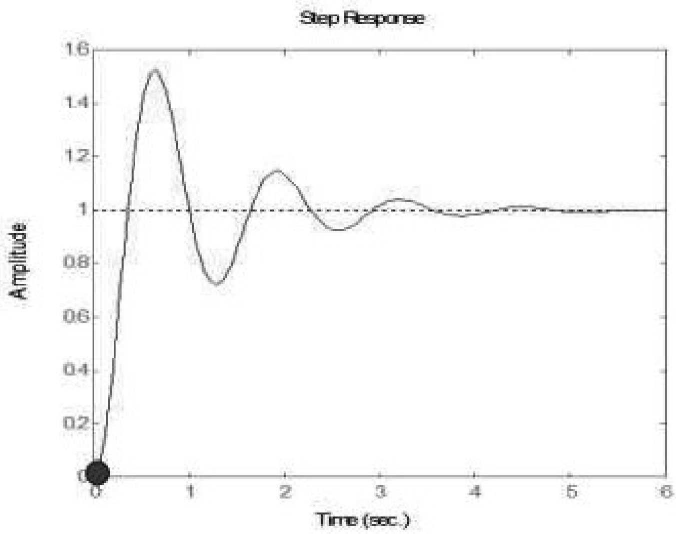
Under Damped System Frequency Response

Bode plot of under damped system



Second Order Response-Under Damped System

$\zeta < 1$ - under damped system . Reaches the final value fast but with overshoot. Less damping more overshoot. Roots are conjugate pairs.



Overshoots to 1.5

Settling time is time Required to reach % of final value Approximately 4.5 sec

3.8 Compensation Techniques

- Performance specifications for the closed-loop system
- Stability
- Transient response: Ts, Ms (settling time, overshoot) or phase and gain margins
- Steady-state response: ess (steady state error)
- Trial and error approach to design