Fundamental of Control Engineering

- Time Response Analysis Lecure-6

Time Response Analysis

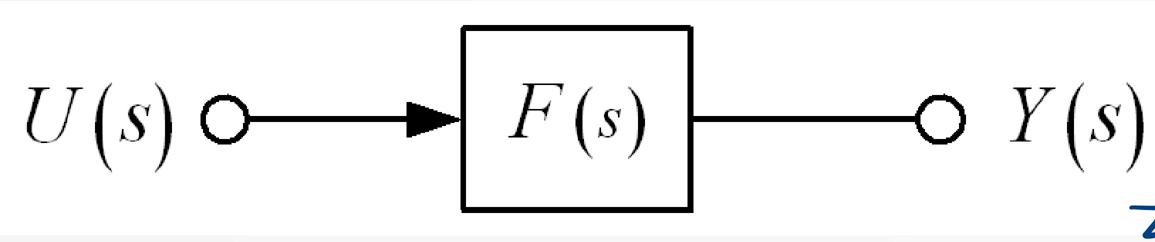
- 1- First step in analysing any control systems is to derive its mathematical model.
- 2- In analyzing and designing any control system we must have a basis of performance comparison with different control systems
- 3- This basis may be setup by specifying particular test input signals and by comparing the responses of various control systems to these input signals.
- 4- System is effected by changing the input test signal or its initial conditions.

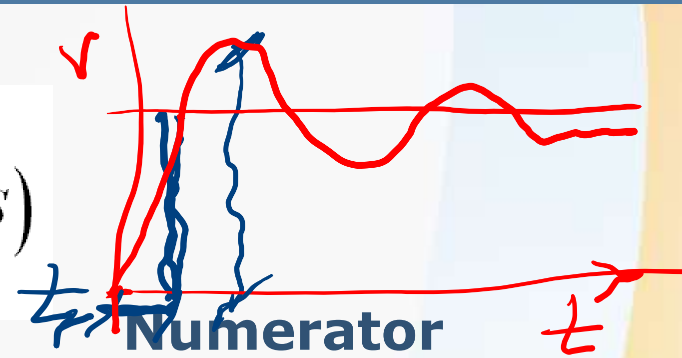
Time Response Analysis

- 5- Typical test signals which commonly used in testing are of the type of: -Step functions - Ramp function - Impulse functions and Sinusoidal functions.
- 6- Time response analysis can be performed only for stable systems.
- 7- Time response of any system consists from Transient response and steady- state response.
- 8- Stability and steady state error are the most important characteristics in any control system.

Definition of Pole and Zero

- Consider the transfer function *F*(*s*):





$$\frac{Y(s)}{U(s)} = F(s) \quad \Rightarrow F(s) = \frac{B(s)}{A(s)}$$

Numerator polynomial

Denominator polynomial

- The system response is given by:

$$Y(s) = F(s)U(s) = \frac{B(s)}{A(s)}U(s)$$

- The **poles** are the values of *s* for which the denominator *A*(*s*) = 0.
- The **zeros** are the values of *s* for which the numerator *B*(*s*) = 0.

Effect of Pole Locations

- Consider the transfer function *F*(*s*):

$$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{s+\sigma} \qquad \Rightarrow Y(s)=\frac{1}{s+\sigma}U(s)$$

A form of first-order transfer function

- The impulse response will be an exponential function:

$$y(t)=e^{-\sigma t}\cdot 1(t)$$

• How?

- When $\sigma>0$, the pole is located at $s<0$,
  ➔ The exponential expression $y(t)$ decays.
  ➔ Impulse response is **stable**.

- When $\sigma<0$, the pole is located at $s>0$,
  ➔ The exponential expression $y(t)$ grows with time.
  ➔ Impulse response is referred to as **unstable**.

Effect of Pole Locations

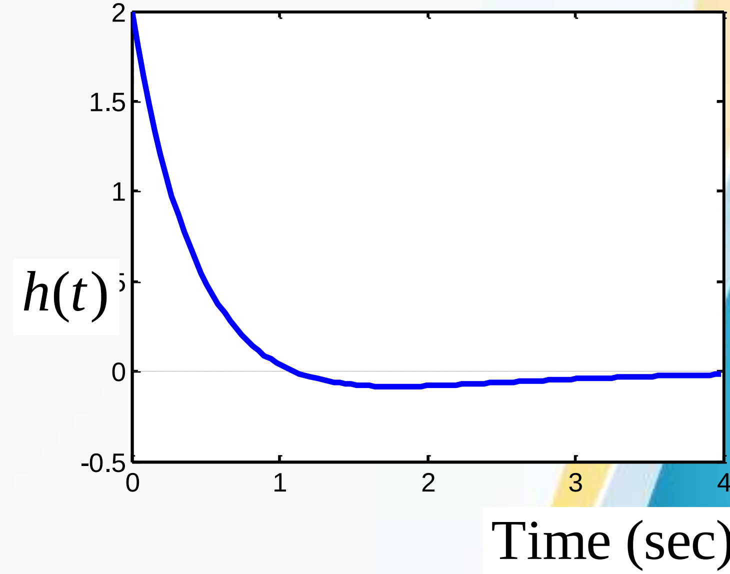
Example:
Find the impulse response of $H(s)$,

$Y(s) = H(s) \cdot U(s)$

$$H(s) = \frac{2s+1}{s^2+3s+2} = \frac{2s+1}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{3}{s+2}$$

$$h(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$h(t) = (-e^{-t} + 3e^{-2t}) \cdot 1(t)$$

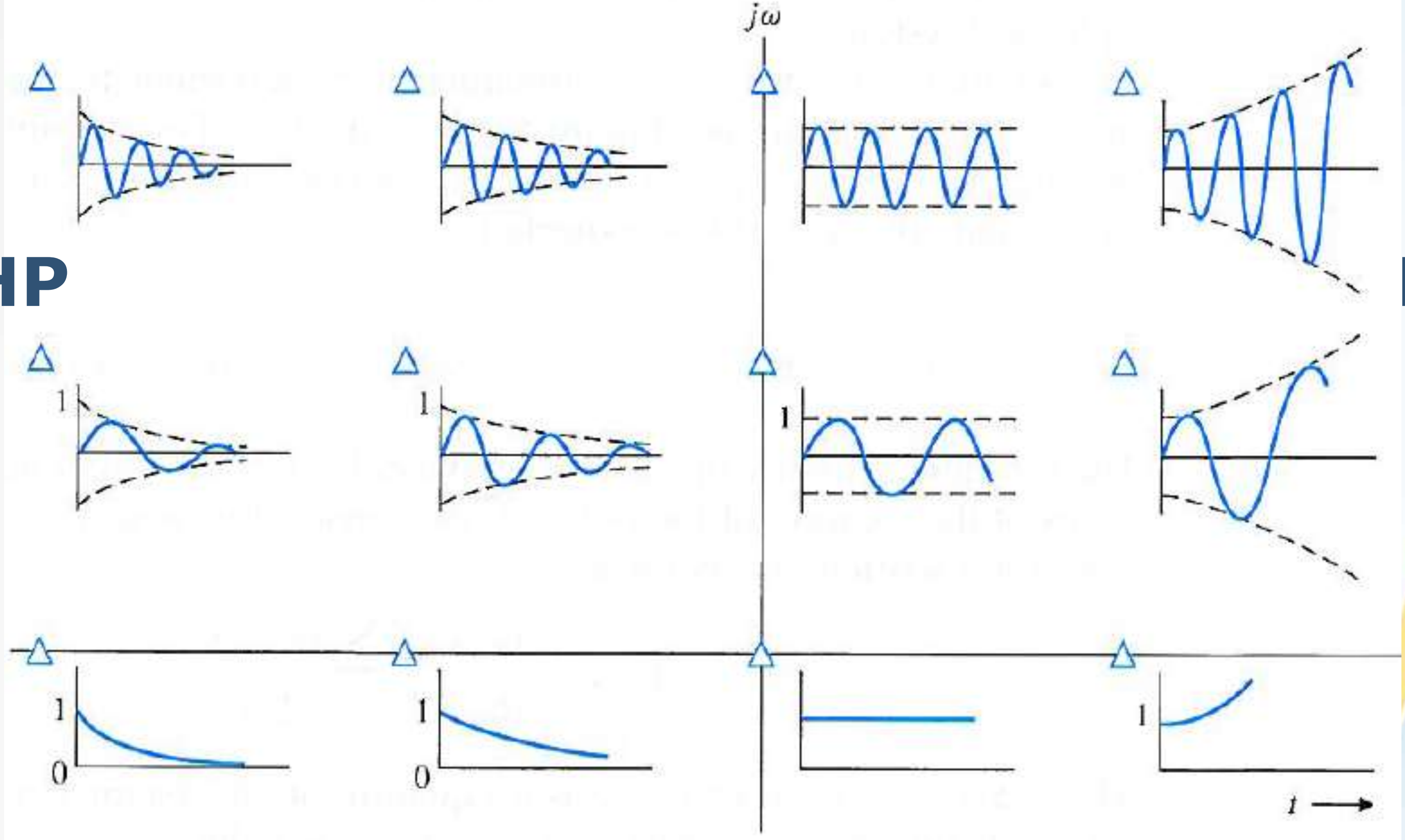


- The terms e^{-t} and e^{-2t}, which are stable, are determined by the poles at $s = -1$ and -2. This is true for more complicated cases as well.
- In general, the response of a transfer function is determined by the locations of its poles.

Effect of Pole Locations

Time function of impulse response assosiated with the pole location in *s*-plane

LHP

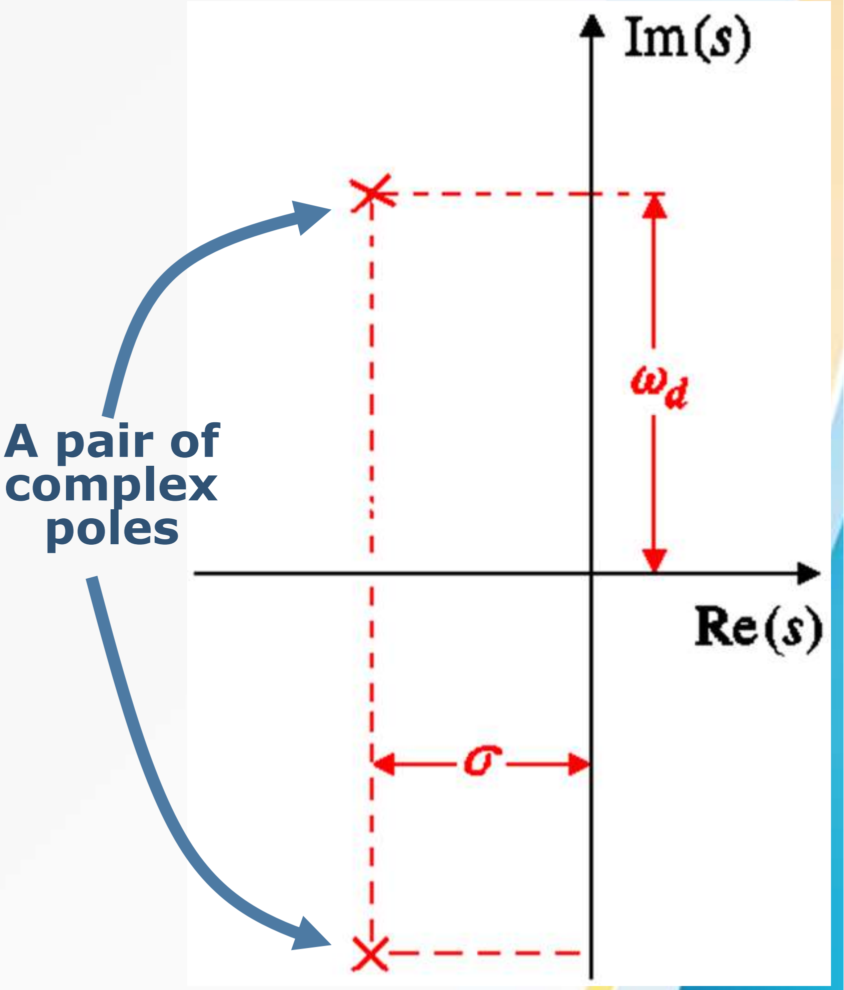


RHP

LHP : left half-plane
RHP : right half-plane

Representation of a Pole in *s*-Domain

- The position of a pole (or a zero) in *s*-domain is defined by its **real** and **imaginary parts,** Re(*s*) and Im(*s*).
- In rectangular coordinates, the complex poles are defined as $(-s \pm j\omega_d)$.
- Complex poles always come in conjugate pairs.



Representation of a Pole in *s*-Domain

- The denominator corresponding to a complex pair will be:

$$A(s) = (s+\sigma - j\omega_d)(s+\sigma + j\omega_d)$$
$$= (s+\sigma)^2 + \omega_d^2$$

$$\frac{\bigcirc}{s^2+3s+2}$$

- On the other hand, the typical polynomial form of a second-order transfer function is:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{s^2+\beta s+k}$$

- Comparing *A*(*s*) and denominator of *H*(*s*), the correspondence between the parameters can be found:

$$\sigma = \zeta\omega_n \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

الجزء الحقيقي من الجذر المعقد

الجزء الخيالي من الجذر المعقد

ζ : damping ratio
ω_n : undamped natural frequency
ω_d : damped frequency

Example1

$$\frac{Y(s)}{X(s)} = \frac{6}{s^2+5s+6} = H(s)$$

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n^2 = 6 \Rightarrow \omega_n = \sqrt{6}$$ = undamped natural Frequency

$$2\zeta\omega_n = 5 \Rightarrow \zeta = \frac{5}{2\omega_n} \approx 1.3$$ =Damping ratio

اذا كان معامل نسبة امتصاص الذبذبة Zeta اكبر من 1 وهنا 1.3 فمعناه
الجذور حقيقيه بدون حد خيالي ولا توجد ذبذبة في الاستجابة

ان معادلة مقام الدالة الانتقالية(H(s Transfer Function تسمى دالة
الخواص Characteristics Equation
وعندما تكون من الدرجة الثانية اي ان اعلى اس لل S فيها هو 2 فيمكن
استخدام قانون الدستورلايجاد الجذرين s1 and s2

$$s_1, s_2 = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$$

$a S^2 + bS + c \quad = \quad S^2 + 4S + 13$

a=1 b=4 c=13

$$s_1 = -2 + 3j$$
$$s_2 = -2 - 3j$$

$$= \frac{-4 \mp \sqrt{16-52}}{2}$$

$$= \frac{-4}{2} \mp \frac{\sqrt{-36}}{2}$$

$$= -2 \mp j3$$

($2 = \sigma$, $3 = \omega_d$)

$$\omega_n = \sqrt{13} = 3.6$$

$$\sigma = \zeta\omega_n = 2$$

$$\omega_d = 3$$

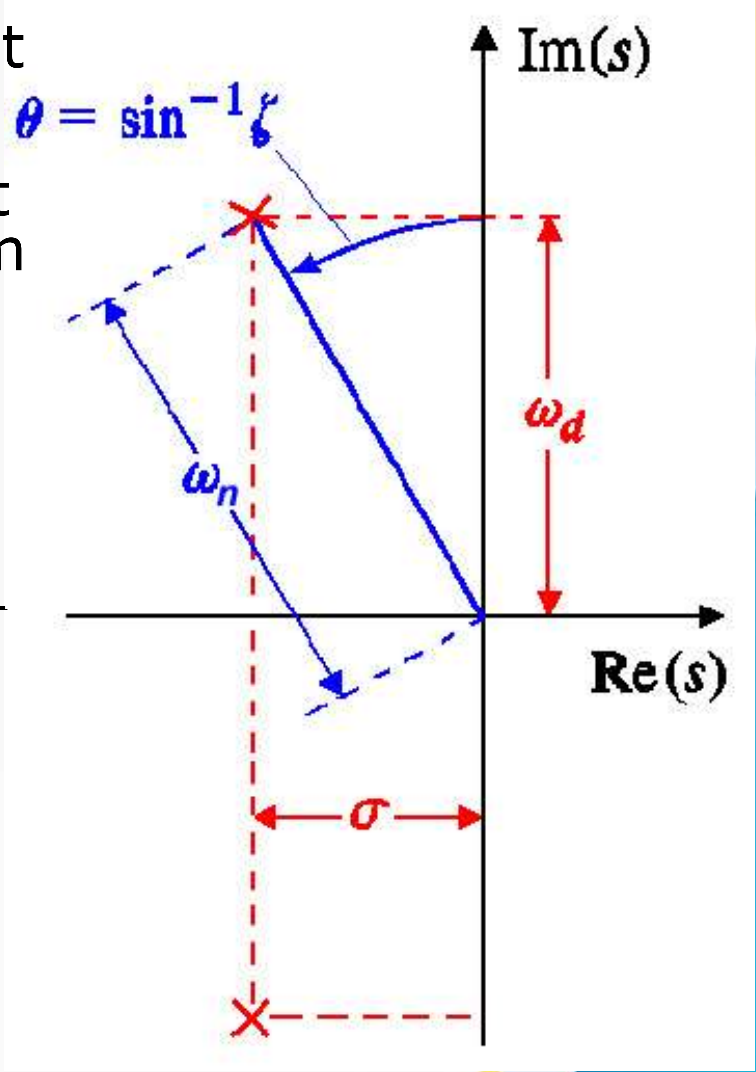
$$\therefore \zeta = \frac{2}{3.6} = 0.55$$

Zeta =0.55 less than 1 then roots must be complex
s1= -2 + 3j and s2 = -2 - 3j

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= 3.6\sqrt{1-0.55^2} = 3.6 \times \sqrt{0.7}$$
$$\simeq 3$$

Representation of a Pole in *s*-Domain

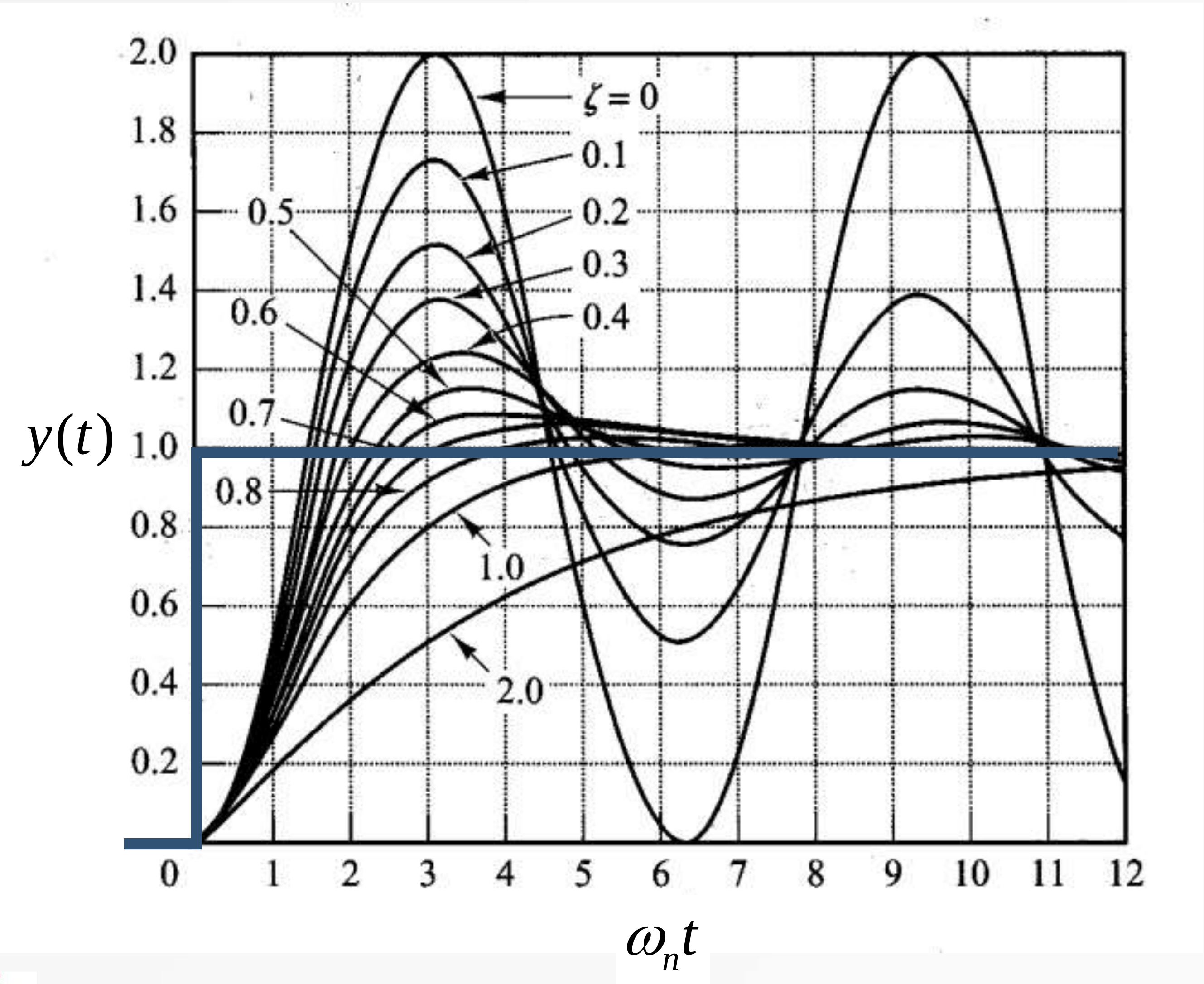
- Previously, in <u>rectangular coordinates</u>, the complex poles are at $(-\sigma \pm j\omega_d)$.
- In <u>polar coordinates</u>, the poles are at $(\omega_n, \sin^{-1}\zeta)$, as can be examined from the figure.



$$\sigma = \zeta\omega_n \quad \Leftrightarrow \quad \zeta = \frac{\sigma}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \Leftrightarrow \quad \omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

Unit Step Resonses of Second-Order System



Effect of Pole Locations

Example:

Find the correlation between the poles and the impulse response of the following system, and further find the exact impulse response.

$$H(s)=\frac{2s+1}{s^2+2s+5}$$

Since $H(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $\omega_n^2=5\Rightarrow\omega_n=\sqrt{5}=2.24\ \text{rad/sec}$

$$2\zeta\omega_n=2\Rightarrow\zeta=0.447$$

The exact response can be otained from:

$$H(s)=\frac{2s+1}{s^2+2s+5}=\frac{2s+1}{(s+1)^2+2^2}\Rightarrow \text{poles at } s=-1\pm j2$$

Effect of Pole Locations

To find the inverse Laplace transform, the righthand side of the last equation is broken into two parts:

$$H(s)=\frac{2s+1}{(s+1)^2+2^2}$$

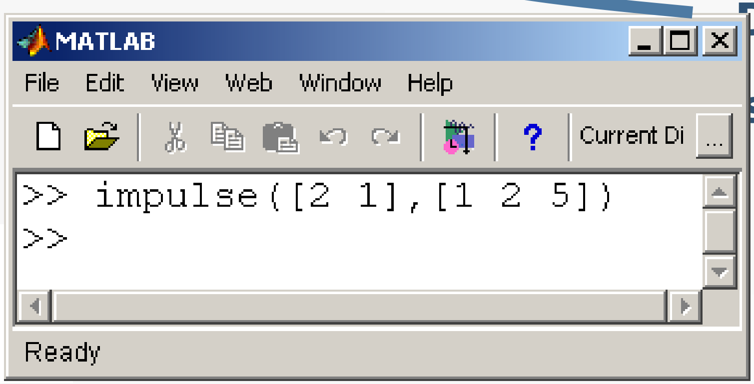
$$=2\frac{s+1}{(s+1)^2+2^2}-\frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

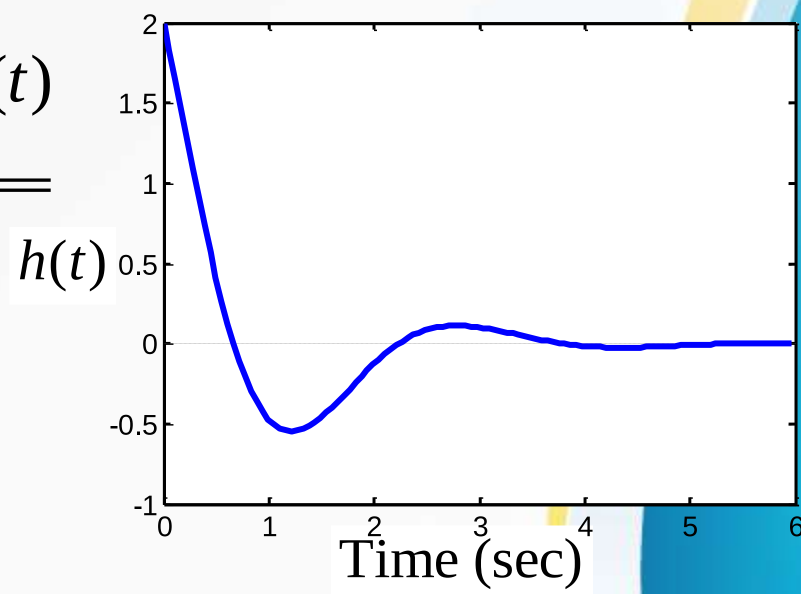
$$h(t)=\mathcal{L}^{-1}\left(H(s)\right)$$

$$=\left(2e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t\right)\cdot 1(t)$$

$f(t)$	$F(s)$
$\sin\omega t\ 1(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t\ 1(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t\ 1(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t\ 1(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Damped nusoidal scillation



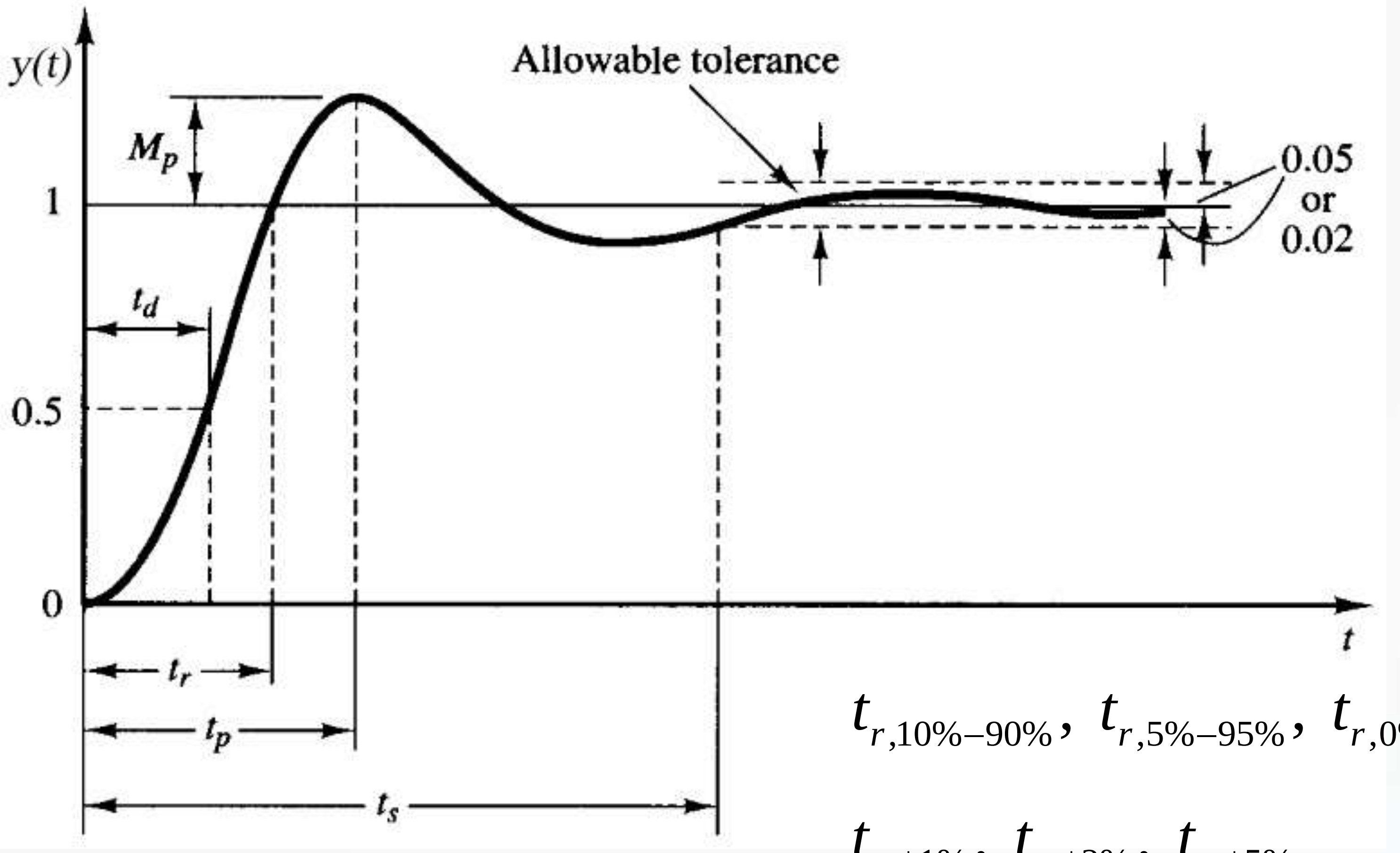


Time Domain Specifications

Specification for a control system design often involve certain requirements associated with the step response of the system:

1. **Delay time**, t_d, is the time required for the response to reach half the final value for the very first time.
2. **Rise time**, t_r, is the time needed by the system to reach the vicinity of its new set point.
3. **Settling time**, t_s, is the time required for the response curve to reach and stay within a range about the final value, of size specified by absolute percentage of the final value.
4. **Overshoot**, M_p, is the maximum peak value of the response measured from the final steady-state value of the response (often expressed as a percentage).
5. **Peak time**, t_p, is the time required for the response to reach the first peak of the overshoot.

Time Domain Specifications



$$t_{r,10\%-90\%},\ t_{r,5\%-95\%},\ t_{r,0\%-100\%}$$

$$t_{s,\pm1\%},\ t_{s,\pm2\%},\ t_{s,\pm5\%}$$

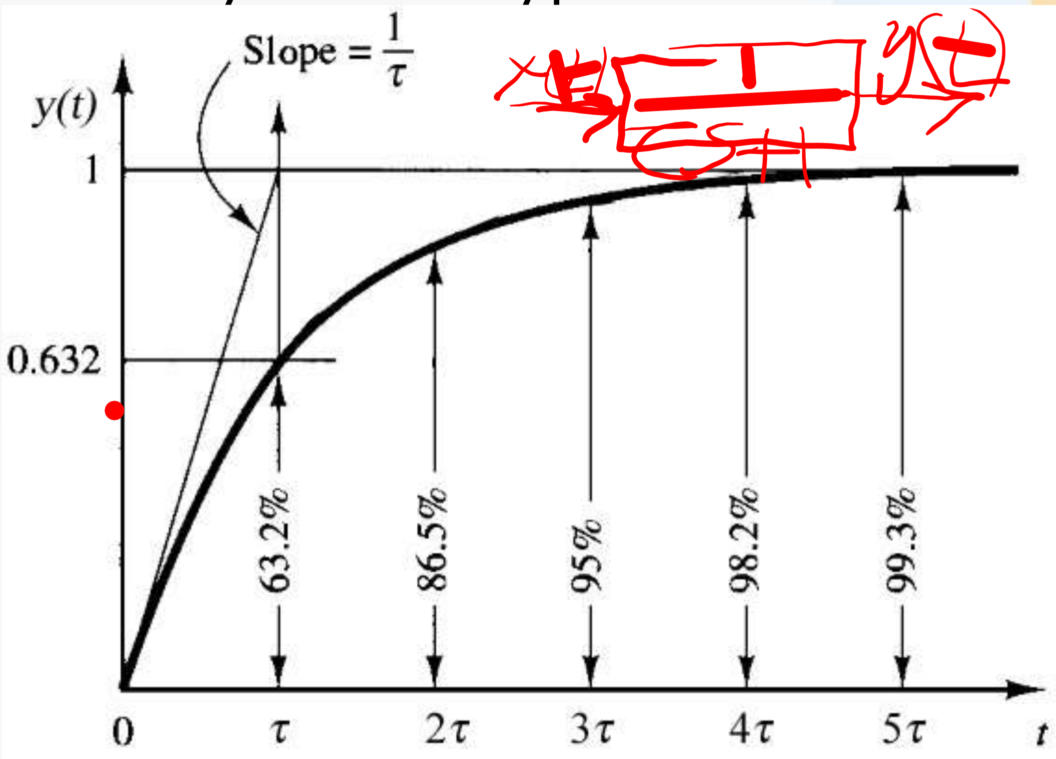
$$\%M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \cdot 100\%$$

First-Order System

- The step response of first-order system in typical form:

$$H(s) = \frac{1}{\tau s + 1}$$

is given by:



$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{1}{s + (1/\tau)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s + (1/\tau)}\right\}$$

$$y(t) = (1 - e^{-t/\tau}) \cdot 1(t)$$

- τ **: time constant**
- **For first order system, M_p and t_p do not apply**

$$e^{-t/\tau} = e^{-1} \quad (t \to \tau)$$

$$e \simeq 2.73$$

$$e^{-1} = \frac{1}{e} = \frac{1}{2.73}$$

$$\simeq 0.364$$

Second-Order System

- The step response of second-order system in typical form:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is given by:

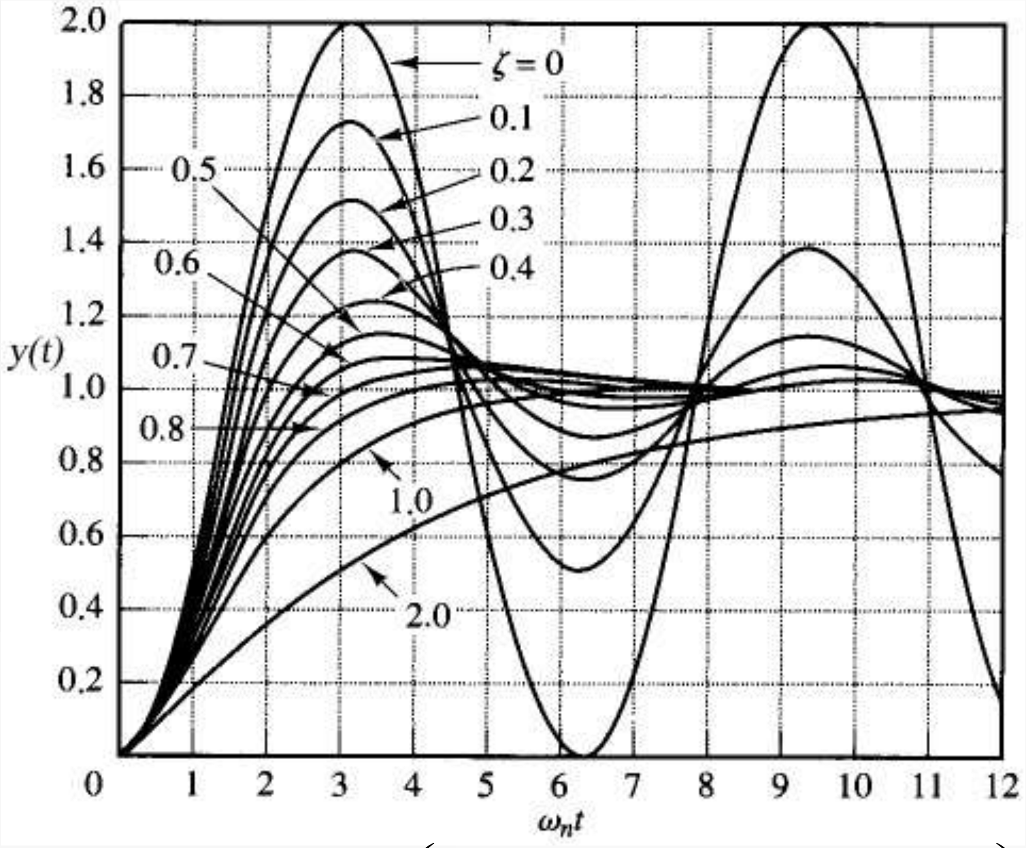
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

Second-Order System



$$y(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

- Time domain specification parameters apply for most second-order systems.
- Exception: overdamped systems, where ζ > 1 (system response similar to first-order system).
- Desirable response of a second-order system is usually acquired with $0.4 < \zeta < 0.8$.

Rise Time

- The step response expression of the second order system is now used to calculate the rise time, $t_{r,0\%-100\%}$:

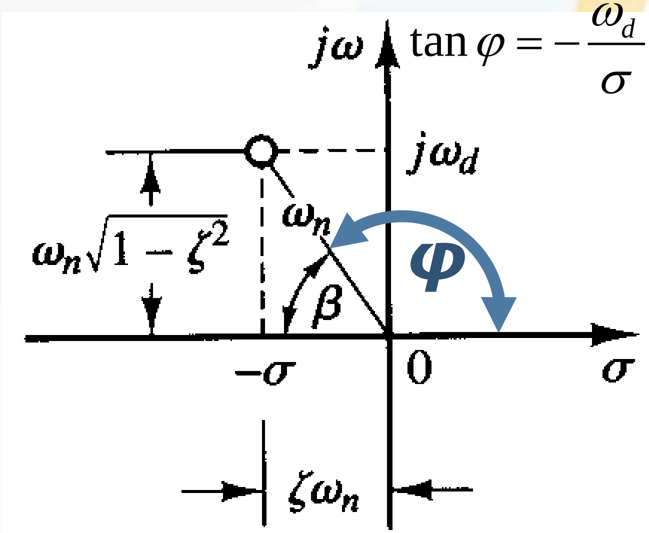
$$y(t_r) = 1 \equiv 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

- Since $e^{-\zeta\omega_n t_r} \neq 0$, this condition will be fulfilled if:

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

or,

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$



$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right) = \frac{(\pi-\beta)}{\omega_d}$$

$$\sigma = \zeta\omega_n$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

t_r is a function of ω_d

Settling Time

- Using the following rule:

$$A\sin\alpha + B\cos\alpha = C\cos(\alpha-\beta),$$

with:

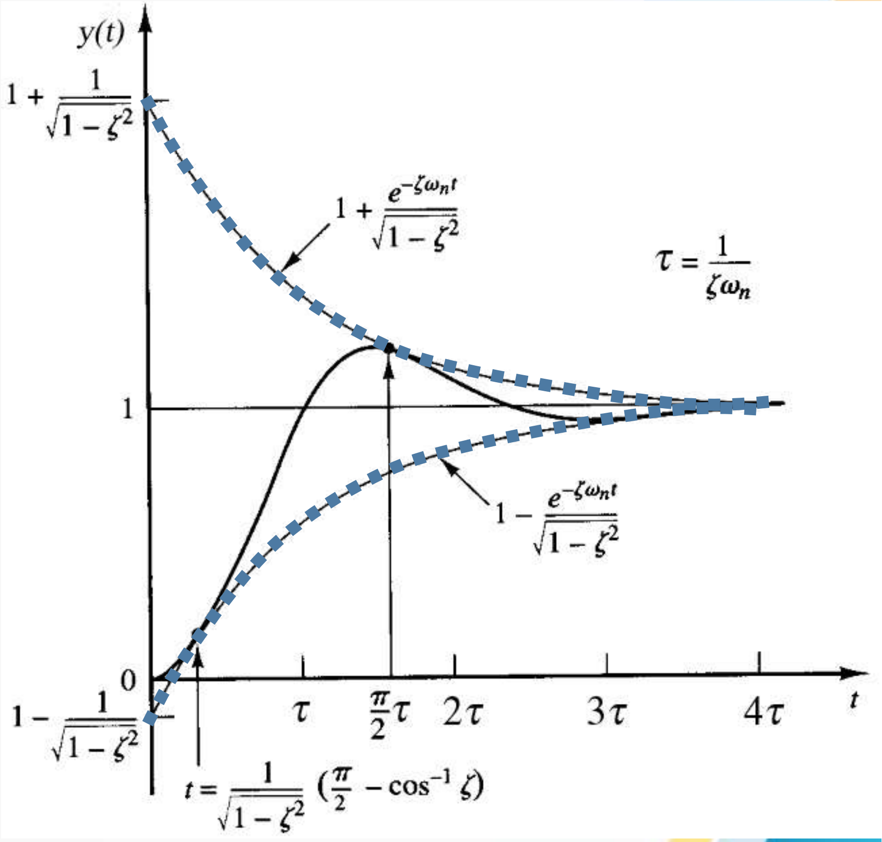
$$C=\sqrt{A^2+B^2},\ \beta=\tan^{-1}\left(\frac{A}{B}\right)$$

- The step response expression can be rewritten as:

$$y(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cdot\left(\cos(\omega_d t-\beta)\right)$$

where:

$$\beta=\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$



t_s **is a function of ζ**

$$\tau=\frac{1}{\zeta\omega_n}$$

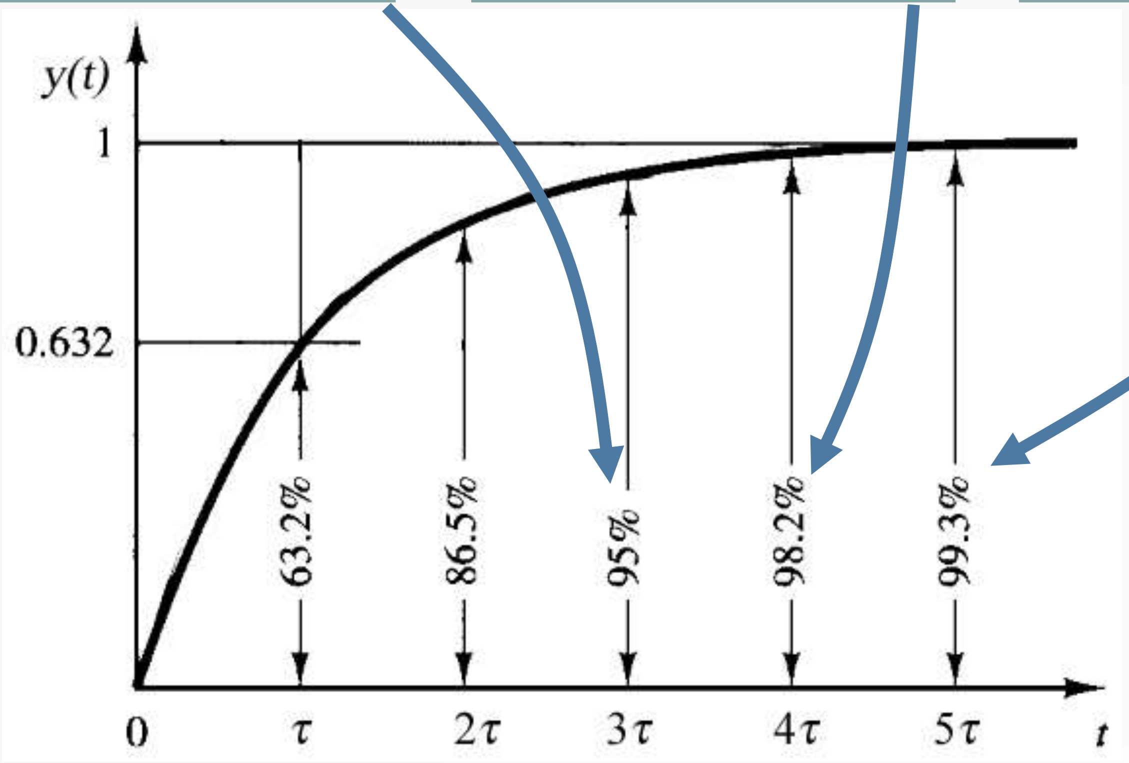
Settling Time

- The time constant of the envelope curves shown previously is $1/\zeta\omega_n$, so that the settling time corresponding to a certain tolerance band may be measured in term of this time constant.

$$t_{s,\pm 5\%} = 3\tau = \frac{3}{\zeta\omega_n}$$

$$t_{s,\pm 2\%} = 4\tau = \frac{4}{\zeta\omega_n}$$

$$t_{s,\pm 1\%} = 5\tau = \frac{5}{\zeta\omega_n}$$



Peak Time

- When the step response $y(t)$ reaches its maximum value (maximum overshoot), its derivative will be zero:

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$y'(t) = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) +$$

$$e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

$$y'(t) = e^{-\zeta\omega_n t}\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right)\sin\omega_d t$$

Peak Time

- At the peak time,

$$y'(t_p) = e^{-\zeta\omega_n t_p}\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right)\underbrace{\sin\omega_d t_p}_{\equiv 0} \equiv 0$$

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \cdots$$

- Since the peak time corresponds to the first peak overshoot,

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

t_p is a function of ω_d

Maximum Overshoot

- Substituting the value of t_p into the expression for $y(t)$,

$$y(t_p)=1-e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right)$$

$$y(t_p)=1-e^{-\zeta\omega_n\cdot\pi/\omega_d}\left(\cos\pi+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)=1+e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$M_p=y(t_p)-y(\infty)$$
$$=(1+e^{-\zeta\pi/\sqrt{1-\zeta^2}})-1$$

$$M_p=e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

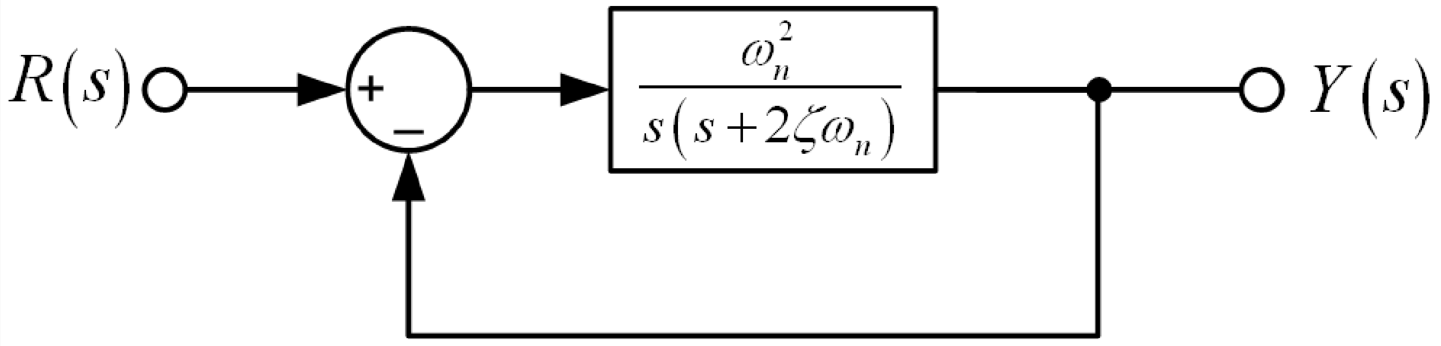
$$\%M_p=\frac{y(t_p)-y(\infty)}{y(\infty)}\cdot 100\%$$

$$\%M_p=e^{-\zeta\pi/\sqrt{1-\zeta^2}}\cdot 100\%$$

if $y(\infty)$ = 1

Example 1: Time Domain Specifications

Example:
Consider a system shown below with $\zeta = 0.6$ and $\omega_n = 5$ rad/s. Obtain the rise time, peak time, maximum overshoot, and settling time of the system when it is subjected to a unit step input.



After block diagram simplification,

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard form of second-order system

Example 1: Time Domain Specifications

$$\zeta = 0.6, \omega_n = 5 \text{ rad/s} \Rightarrow \omega_d = \sqrt{1-\zeta^2}\omega_n = \sqrt{1-0.6^2}\cdot 5 = 4 \text{ rad/s}$$

$$\Rightarrow \sigma = \zeta\omega_n = 0.6\cdot 5 = 3 \text{ rad/s}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$$

In second quadrant

$$= \frac{1}{4}\tan^{-1}\left(-\frac{4}{3}\right) = \frac{1}{4}(\pi - 0.927) = \underline{\underline{0.554 \text{ s}}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = \underline{\underline{0.785 \text{ s}}}$$

Example 1: Time Domain Specifications

$$M_p = y(t_p) - y(\infty) = (1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}) - 1$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = e^{-(0.6\cdot\pi)/0.8} = \underline{\underline{0.0948}}$$

$$\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \cdot 100\% = \underline{\underline{9.48\%}}$$

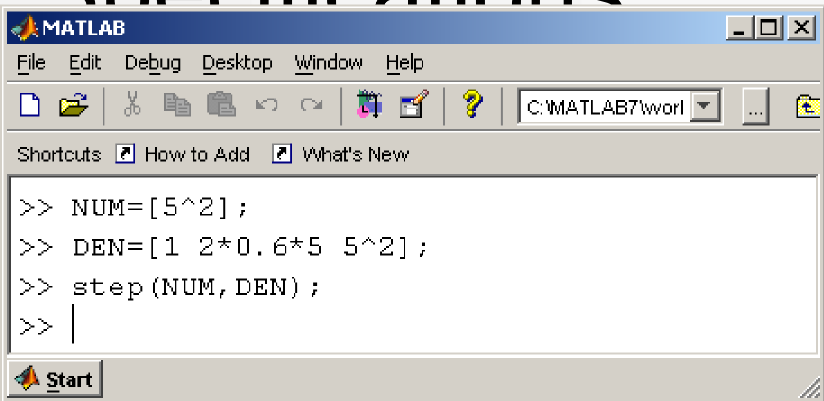
$$t_{s,\pm 2\%} = \frac{4}{\zeta\omega_n} = \frac{4}{0.6\cdot 5} = \underline{\underline{1.333\ \text{s}}}$$

$$t_{s,\pm 5\%} = \frac{3}{\zeta\omega_n} = \frac{3}{0.6\cdot 5} = \underline{\underline{1\ \text{s}}}$$

Check $y(\infty)$ for unit step input, if

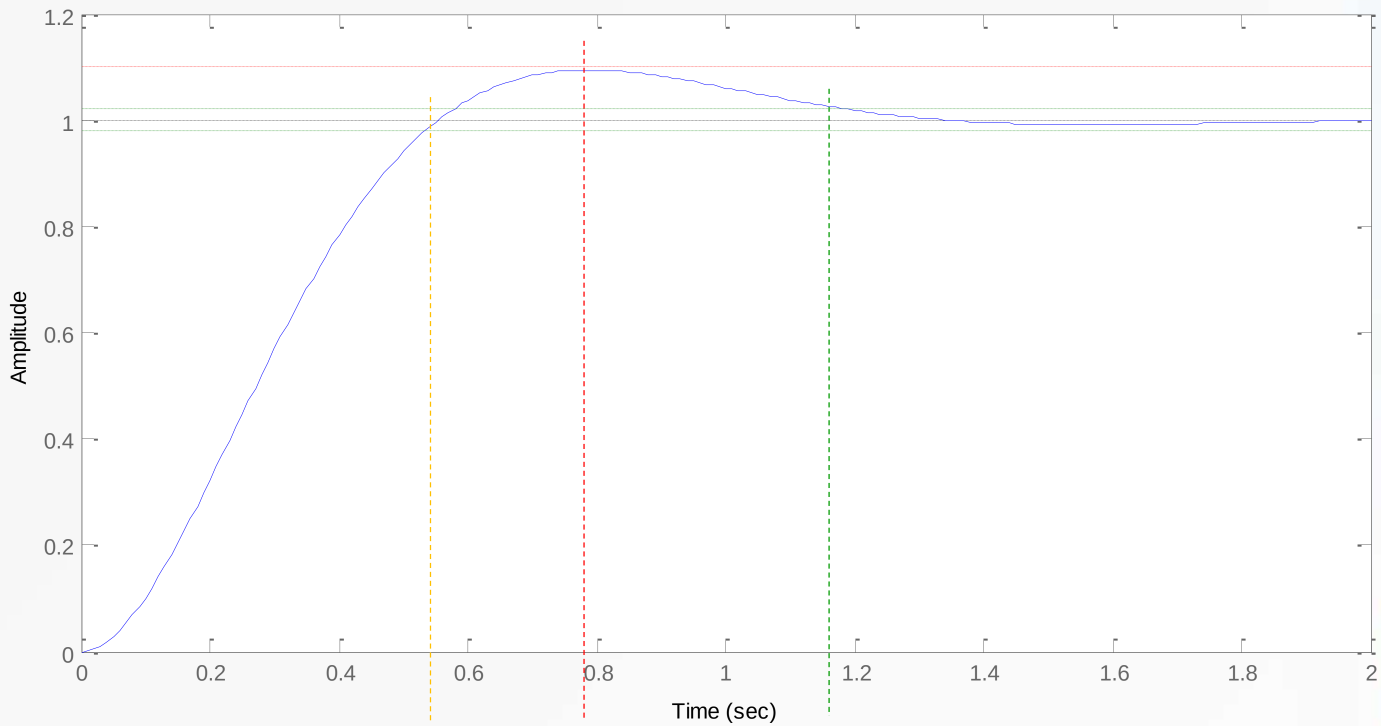
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example 1: Time Domain Specifications



$$t_r = 0.554 \text{ s}, \; t_p = 0.785 \text{ s}$$

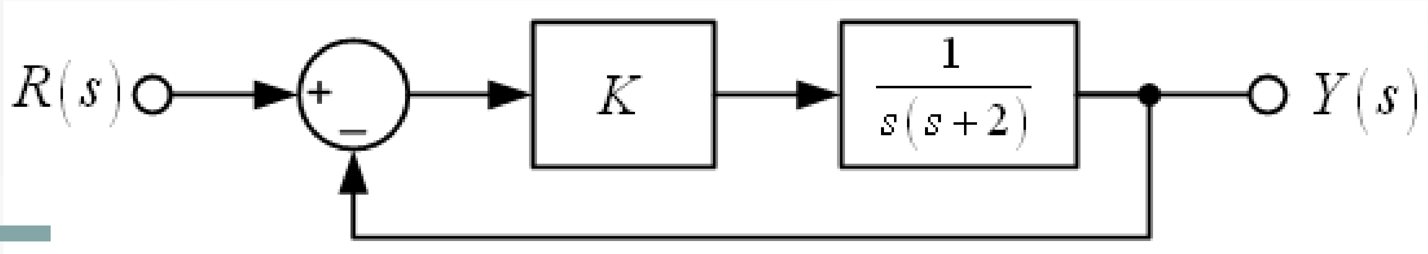
$$M_p = 9.48\%, t_s = 1.333 \text{ s}$$



Example 2: Time Domain Specifications

Example:

For the unity feedback system shown below, specify the gain *K* of the proportional controller so that the output *y*(*t*) has an overshoot of no more than 10% in response to a unit step.



$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s^2+2s+K}$$

$$\Rightarrow 2\zeta\omega_n = 2$$

$$\Rightarrow \omega_n^2 = K$$

$$\%M_p \le 10\% \Rightarrow e^{-\zeta\pi/\sqrt{1-\zeta^2}} \le 0.1 \quad \Rightarrow \zeta \ge 0.592$$

$$\Rightarrow \omega_n = \frac{1}{\zeta} \le \frac{1}{0.592} = 1.689$$

$$\Rightarrow K = \omega_n^2 \le 1.689^2 = 2.853$$

$$\therefore \underline{\underline{0 < K \le 2.853}}$$

Example 2: Time Domain Specifications