Middle Technical University
Electrical Engineering Technical College
Department of Medical Instrumentation Engineering Techniques

Control System

1. **COURSE Title: Control System**
2. **Instructor: Assist. Prof. Dr. Ahmed R. Ajel**
3. **Text book**
   - Modern Control Systems, R. C. Dorf and R. H. Bishop, Pearson Prentice Hall, 11th edition, 2008.
   - Modern Control Engineering, Ogata Katsuhiko, 5th Edition, Prentice-Hall, 2010.
   - Control System Engineering, Norman S. Nise, California State Polytechnic University, Pomona, John Wiley & Sons, Inc. ,6th_Edition, 2011.

January 2022

Electrical Engineering Technical College	30 Weeks	No. of week hours		
Department of Medical Instrumentation Engineering Techniques		Th.	Pr.	Unit
		2	2	6
Fourth Year	Subject: Control System.			

التطبيقية . **أهداف المادة :** التعرف على مكونات دوائر السيطرة و أنواع المسيطرات و استخداماتها ودوائرها

Week	Syllabus
1st	Introduction to linear control engineering.
2nd , 3rd	Mathematical background; Laplace transform, complex variable, matrices.
4th , 5th , 6th	Transfer function, block diagram representation and reduction, signal flow diagram.
7th , 8th , 9th	Time domain analysis, steady – state transient analysis.
10th , 11th	Stability analysis; Routh, Nyquist.
12th , 13th	Root locus technique.
14th , 15th , 16th	Frequency domain analysis, Gain margin, phase margin and bode plot.
17th , 18th	Frequency domain synthesis, phase lead.
19th , 20th	Compensation, phase – lag compensation lag – lead compensation.
21st , 22nd , 23r, 24th	PID controllers design.
25th, 26th , 27th	State space representation and analysis.
28th , 29th	State diagram; analogue computer.
30th	Block diagram representation.

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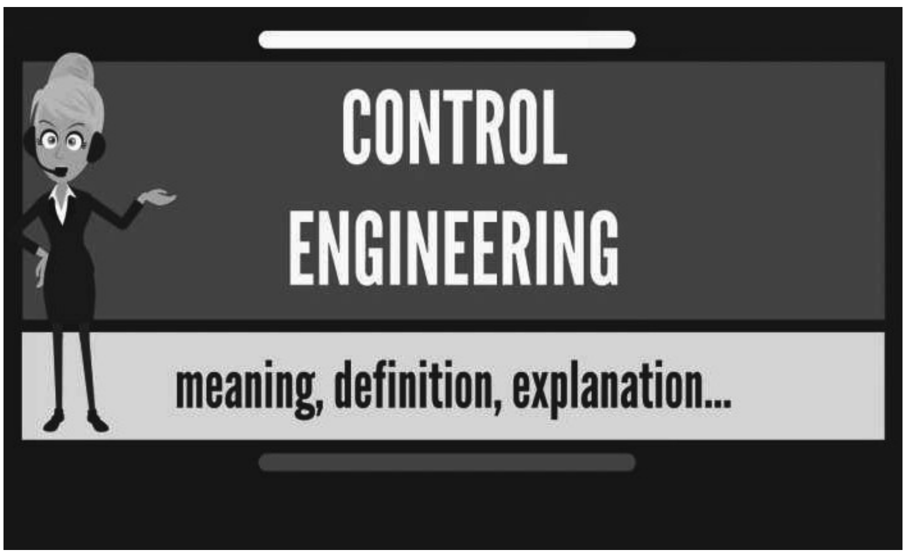
- ***Lecture 1 (1st Week)***

Introduction to Linear Control Engineering



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** A Control Systems Engineer is responsible for designing, developing, and implementing solutions that control dynamic systems.

c. **Central Ideas:** Control Systems Engineering is the engineering approach taken to understand how the process can be managed by automation devices and to implement such into operation.

d. **Objectives:** After completing this lecture, the student will be able to:

1. Define a control system and describe some applications.
2. Describe historical developments leading to modern day control theory
3. Describe the basic features and configurations of control systems
4. Describe control systems analysis and design objectives
5. Describe a control system's design process
6. Describe the benefit from studying control systems

2. Pre-Test:

1. What is control system?
2. What are the various types of 'control system'?
3. The key elements of a control process include a characteristic to be tested, sensors, comparative standards, and implementation. State True or false
4. Feedback control systems are referred to as closed loop systems. State True or false.
5. What is the effect of feedback in the overall gain of the system?

   a) increases b) decreases c) zero d) no change

Note: Check your answers in "Answer Keys" in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

3. Theory:

3.1 Introduction:

Control systems are an integral part of modern society. Numerous applications are all around us: The rockets fire, and the space shuttle lifts off to earth orbit; in splashing cooling water, a metallic part is automatically machined; a self-guided vehicle delivering material to workstations in an aerospace assembly plant glides along the floor seeking its destination. These are just a few examples of the automatically controlled systems that can create. the only creators of automatically controlled systems; these systems also exist in nature. Within our own bodies are numerous control systems, such as the pancreas, which regulates our blood sugar. In time of ‘‘fight or flight,’’ our adrenaline increases along with our heart rate, causing more oxygen to be delivered to our cells. Our eyes follow a moving object to keep it in view; our hands grasp the object and place it precisely at a predetermined location. Even the nonphysical world appears to be automatically regulated. Models have been suggested showing automatic control of student performance. The input to the model is the student's available study time, and the output is the grade. The model can be used to predict the time required for the grade to rise if a sudden increase in study time is available. Using this model, you can determine whether increased study is worth the effort during the last week of the term.

3.2 Control System Definition

A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input. Figure 1.1 shows a control system in its simplest form, where the input represents a desired output. For example, consider an elevator. When the fourth-floor button is pressed on the first floor, the elevator rises to the fourth floor with a speed and floor-leveling accuracy designed for passenger comfort. The push of the fourth-floor button is an input that represents our desired output, shown as a step function in Figure 1.2. The performance of the elevator can be seen from the elevator response curve in the figure. Two major measures of performance are apparent: (1) the transient response and (2) the steady-state error. In our example, passenger comfort and passenger patience are dependent upon the transient response. If this response is too fast, passenger comfort is sacrificed; if too slow, passenger patience is sacrificed. The steady-state error is another important performance specification since passenger safety and convenience would be sacrificed if the elevator did not properly level.

FIG. 1: Simplified description of a control system.

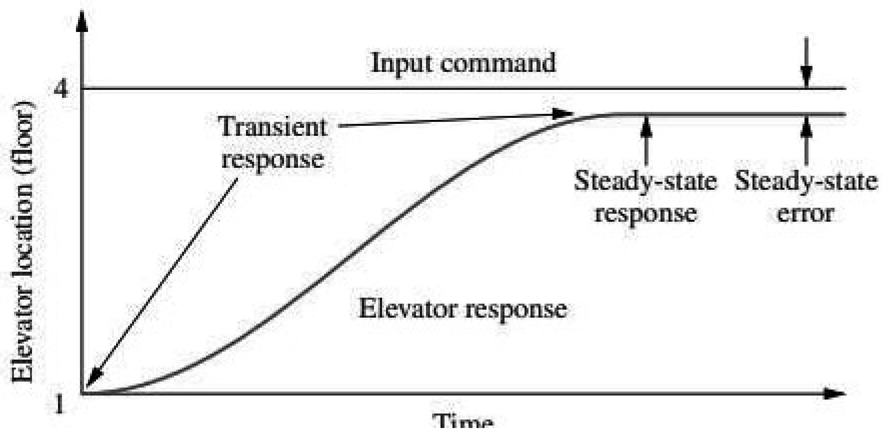


FIG. 2: Elevator response.

3.2 Advantages of Control Systems

With control systems we can move large equipment with precision that would otherwise be impossible. We can point huge antennas toward the farthest reaches of the universe to pick up faint radio signals; controlling these antennas by hand would be impossible. Because of control systems, elevators carry us quickly to our destination, automatically stopping at the right floor.

We alone could not provide the power required for the load and the speed; motors provide the power, and control systems regulate the position and speed. We build control systems for four primary reasons:

1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation for disturbances

For example, a radar antenna, positioned by the low-power rotation of a knob at the input, requires a large amount of power for its output rotation. A control system can produce the needed power amplification, or power gain.

3.3 A History of Control Systems

1. Liquid-Level Control

The Greeks began engineering feedback systems around 300 B.C. Awater clock invented by Ktesibios operated by having water trickle into a measuring container at a constant rate. The level of water in the measuring container could be used to tell time

2. Steam Pressure and Temperature Controls

Regulation of steam pressure began around1681with Denis Papin's invention of the safety valve. The concept was further elaborated on by weighting the valve top. If the upward pressure from the boiler exceeded the weight, steam was released, and the pressure decreased. If it did not exceed the weight, the valve did not open, and the pressure inside the boiler increased.

3. Speed Control

In 1745, speed control was applied to a windmill by Edmund Lee. Increasing winds pitched the blades farther back, so that less area was available decreased, more blade area was available. William Cubitt improved on the idea in 1809 by dividing the windmill sail into movable louvers.

4. Camera laser system for medical purpose as shown in Figure 3

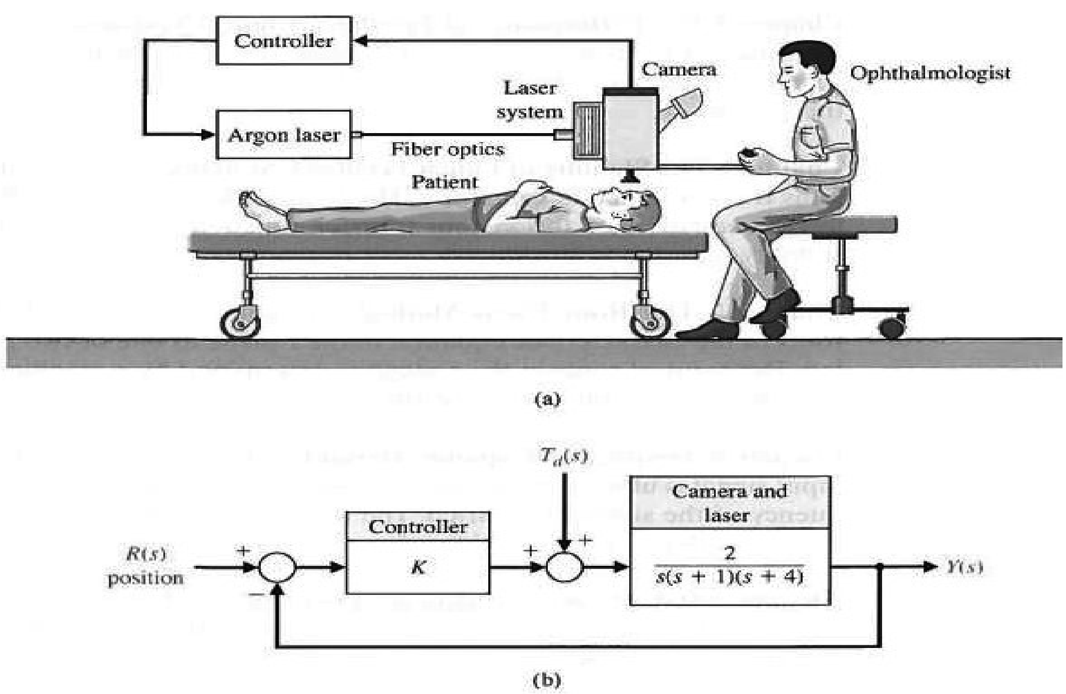


FIG. 3: Camera laser system for medical purpose

4.4 Definitions:

Processes. The *Merriam–Webster Dictionary* defines a process to be a natural, progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead toward a particular result or end; or an artifical or voluntary, progressively continuing operation that consists of a series of controlled actions or movements systematically directed toward a particular result or end. In this book we shall call any operation to be controlled a *process*. Examples are chemical, economic, and biological processes.

Systems. A system is a combination of components that act together and perform a certain objective. A system is not limited to physical ones. The concept of the system can be applied to abstract, dynamic phenomena such as those encountered in economics. The word system should, therefore, be interpreted to imply physical, biological, economic, and the like, systems.

Disturbances. A disturbance is a signal that tends to adversely affect the value of the output of a system. If a disturbance is generated within the system, it is called *internal*, while an *external* disturbance is generated outside the system and is an input.

Feedback Control. Feedback control refers to an operation that, in the presence of disturbances, tends to reduce the difference between the output of a system and some reference input and that does so on the basis of this difference. Here only unpredictable disturbances are so specified, since predictable or known disturbances can always be compensated for within the system.

4.5 Closed-Loop Control Versus Open-Loop Control:

Closed-Loop Control Systems. Feedback control systems are often referred to as closed-loop control systems. In practice, the terms feedback control and closed-loop control are used interchangeably. In a closed-loop control system the actuating error signal, which is the difference between the input signal and the feedback signal (which may be the output signal itself or a function of the output signal and its derivatives and/or integrals), is fed to the controller so as to reduce the error and bring the output of the system to a desired value. The term closed-loop control always implies the use of feedback control action in order to reduce system error.

Open-Loop Control Systems. Those systems in which the output has no effect on the control action are called open-loop control systems. In other words, in an open loop control system the output is neither measured nor fed back for comparison with the input. One practical example is a washing machine. Soaking, washing, and rinsing in the washer operate on a time basis. The machine does not measure the output signal, that is, the cleanliness of the clothes, as shown in Figure 4

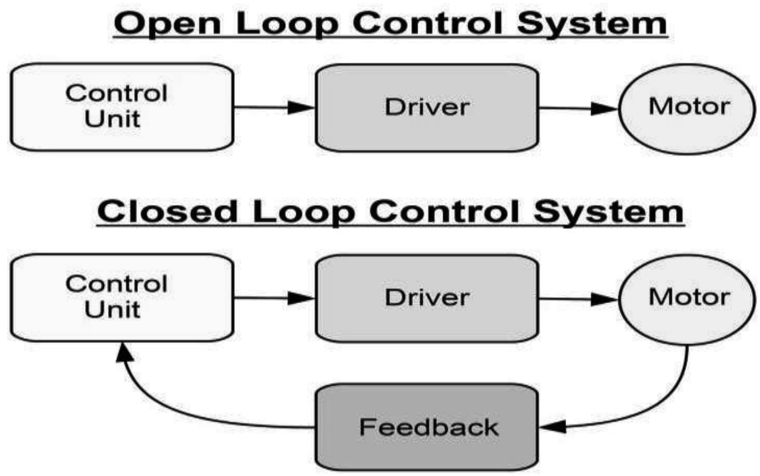


FIG. 4: Simplified block diagram description of open and closed loop control system.

4. Self- Test

1. State the application of control system I medical instrumentation?
2. Compare between open and closed loop system?

4.6 Feedback System Fundamentals

To achieve good control there are typical goals:

• Stability: The system must be stable at all times. This is an absolute requirement.

• Tracking: The system output must track the command reference signal as closely as possible.

• Disturbance rejection: The system output must be as insensitive as possible to disturbance inputs.

• Robustness: The aforementioned goals must be met even if the model used in the design is not completely accurate or if the dynamics of the physical system change over time.

5. Post- Test

1. Define all signal in control system.
2. Controlling is last element of elements of control system. State True or False
3. Draw open and closed loop system?

Answer Keys

Pre- Test

1. Control System is a system in which the output is controlled by varying the input.
2. Control System including:

   Casual systems.

   Linear Time invariant systems.

   Time variant systems.

   Non-linear systems.
3. True.
4. True
5. b. decreases

explanation: -the feedback reduces the overall gain of the system. as soon as we introduce feedback in the system to make the system stable, gain is reduced.

Self-Test

1. see sec. 3.3 and give more examples.
2. See sec. 4.5.

Post- Test

1. See sec. 4.4
2. False.
3. See Fig.4.

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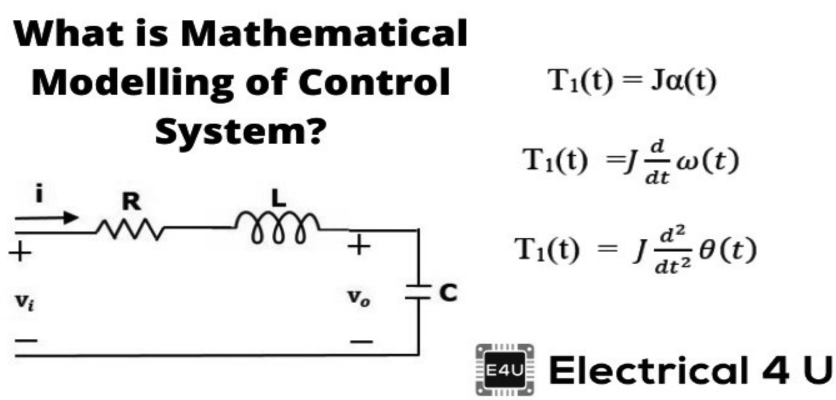
- ***Lecture 2 (2nd and 3rd Weeks)***

Mathematical background; Laplace transform, complex variable, matrices



For

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Medical Instrument Department

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Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** In studying control systems the reader must be able to model dynamic systems in mathematical terms and analyze their dynamic characteristics.

c. **Central Ideas:** A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well.

d. **Objectives:** After completing this lecture, the student will be able to:
   1. Explain how the Laplace transform relates to the transient and sinusoidal responses of a system.
   2. Convert time functions into the Laplace domain.
   3. Use Laplace transforms to convert differential equations into algebraic equations.
   4. Take the Inverse Laplace transform and find the time response of a system.
   5. Use Initial and Final Value Theorems to find the steady state response of a system

2. Pre-Test:

1. What is the mathematical model of control system?

2. A linear system at rest is subject to an input signal r(t)=1-e-t. The response of the system for t>0 is given by c(t)=1-e-2t. The transfer function of the system is?

A. (s+2)/(s+1) B. (s+1)/(s+2) C. 2(s+1)/(s+2) D. (s+1)/2(s+2)

Note: Check your answers in "Answer Keys" in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

3.Theory:

3.1 Introduction: Mathematical Models. Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations. On the other hand, for the transient-response or frequency-response analysis of single-input, single-output, linear, time-invariant systems, the transfer-function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

3.2 Linear Time-Invariant Systems and Linear Time-Varying Systems.

A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant differential equations—that is, constant-coefficient differential equations. Such systems are called linear time-invariant (or linear constant-coefficient) systems. Systems that are represented by differential equations whose coefficients are functions of time are called linear time-varying systems. An example of a time-varying control system is a spacecraft control system. (The mass of a spacecraft changes due to fuel consumption.)

3.3 Transfer Function. The transfer function of a linear, time-invariant

Differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero. Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$

$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

3.4 The Laplace Transform: Basic Definitions and Results

- The given \hard" problem is transformed into a \simple" equation.
- This simple equation is solved by purely algebraic manipulations.
- The solution of the simple equation is transformed back to obtain the solution of the given problem.
- In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration. The above procedure can be summarized by Figure 4.1

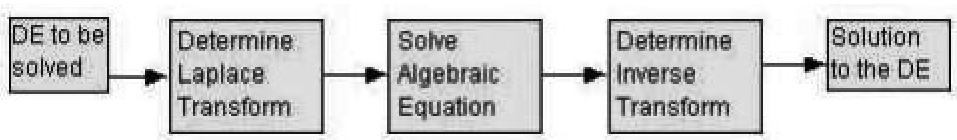


Figure 4.1: Laplace Method.

In this section we introduce the concept of Laplace transform and discuss some of its properties.
The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

Example 3.1

Find the Laplace transform, if it exists, of each of the following functions (a) f(t) = e^{at} (b) f(t) = 1 (c) f(t) = t (d) f(t) = e^{t2}

(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt.$$

But

$$\int_0^T e^{-(s-a)t}dt = \begin{cases} T & \text{if } s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{1}{s}, \ s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}, \ s > 0.$$

(d) Again using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \le 0$ then $t^2 - st \ge 0$ so that $e^{t^2-st} \ge 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \ge \int_0^\infty$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also divergent. Now, if $s > 0$ then $\int_s^\infty e^{t(t-s)}dt \ge \int_s^\infty dt$. By the same reasoning

Example 3.2

Show that any bounded function f(t) for t ≥ 0 is exponentially bounded.

Solution. Since f(t) is bounded for t ≥ 0; there is a positive constant M such that

f(t)j ≤ M for all t ≥ 0: But this is the same as (1) with a = 0 and C = 0: Thus, f(t) has is exponentially bounded Another question that comes to mind is whether it is possible to relax the condition of continuity on the function f(t): Let's look at the following situation.

$$\text{(a) } f(t) = t^n \quad \text{(b) } f(t) = t^n \sin at$$

Solution.
(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, we have $t^n \leq n!e^t$. Hence, t^n is piecewise continuous and exponentially bounded.
(b) Since $|t^n \sin at| \leq n!e^t$, we have $t^n \sin at$ is piecewise continuous and exponentially bounded ■

Next, we would like to establish the existence of the Laplace transform for all functions that are piecewise continuous and have exponential order at infinity. For that purpose we need the following comparison theorem from calculus.

3.5 The Inverse Laplace Transform

Let $f(t)$ and $g(t)$ be two elements in $\mathcal{PE}$ with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

The standard techniques used to prove this theorem(i.e., complex analysis, residue computations, and/or Fourier's integral inversion theorem) are generally beyond the scope of an introductory differential equations course. The interested reader can find a proof in the book "Operational Mathematics" by Ruel Vance Churchill or in D.V. Widder "The Laplace Transform".
With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function f of exponential order at infinity whose Laplace transform is F, we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \Longleftrightarrow F(s) = \mathcal{L}[f(t)].$$

Example 3.3
Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right), \quad s > 1.$

Solution.
From Example 43.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1}, \ s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t, \ t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.
When the function $f(t)$ is not continuous, the uniqueness of the inverse

Table 1: Laplace Transformation

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

3.5 Properties of Laplace Transform

Property 8.1 (Linearity). If $\mathcal{L}\{f(x)\} = F(s)$ and $\mathcal{L}\{g(x)\} = G(s)$, then for any two constants c_1 and c_2

$$\begin{aligned}\mathcal{L}\{c_1 f(x) + c_2 g(x)\} &= c_1 \mathcal{L}\{f(x)\} + c_2 \mathcal{L}\{g(x)\} \\ &= c_1 F(s) + c_2 G(s)\end{aligned}$$

Property 8.2. If $\mathcal{L}\{f(x)\} = F(s)$, then for any constant a

$$\mathcal{L}\{e^{ax} f(x)\} = F(s-a)$$

Property 8.3. If $\mathcal{L}\{f(x)\} = F(s)$, then for any positive integer n

$$\mathcal{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Property 8.4. If $\mathcal{L}\{f(x)\} = F(s)$ and if $\lim_{\substack{x\to 0 \\ x>0}} \frac{f(x)}{x}$ exists, then

$$\mathcal{L}\left\{\frac{1}{x} f(x)\right\} = \int_s^{\infty} F(t)dt$$

Property 8.5. If $\mathcal{L}\{f(x)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^x f(t)dt\right\} = \frac{1}{s} F(s)$$

Property 8.6. If $f(x)$ is periodic with period ω, that is, $f(x+\omega) = f(x)$, then

$$\mathcal{L}\{f(x)\} = \frac{\int_0^{\omega} e^{-sx} f(x)dx}{1-e^{-\omega s}}$$

3.6 Manipulating Denominators

The method of *completing the square* converts a quadratic polynomial into the sum of squares, a form that appears in many of the denominators in the Appendix. In particular, for the quadratic

$$\begin{aligned} as^2 + bs + c &= a\left(s^2 + \frac{b}{a}s\right) + c \\ &= a\left[s^2 + \frac{b}{a}s + \left(\frac{b}{2a}\right)^2\right] + \left[c - \frac{b^2}{4a}\right] \\ &= a\left(s + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a(s+k)^2 + h^2 \end{aligned}$$

where $k = b/2a$ and $h = \sqrt{c-(b^2/4a)}$.

The method of *partial fractions* transforms a function of the form $a(s)/b(s)$, where both $a(s)$ and $b(s)$ are polynomials in s, into the sum of other fractions such that the denominator of each new fraction is either a first-degree or a quadratic polynomial raised to some power. The method requires only that the degree of $a(s)$ be less than the degree of $b(s)$ (if this is not the case, first perform long division, and consider the remainder term) and $b(s)$ be factored into the product of distinct linear and quadratic polynomials raised to various powers.

The method is carried out as follows. To each factor of $b(s)$ of the form $(s-a)^m$, assign a sum of m fractions, of the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}$$

To each factor of $b(s)$ of the form $(s^2+bs+c)^p$, assign a sum of p fractions, of the form

$$\frac{B_1 s + C_1}{s^2+bs+c} + \frac{B_2 s + C_2}{(s^2+bs+c)^2} + \cdots + \frac{B_p s + C_p}{(s^2+bs+c)^p}$$

4. Self- Test

1. Find $\mathcal{L}\, e^{ax}$
2. Find $\mathcal{L}^{-1}\left\{\frac{S+3}{(S-2)(S+1)}\right\}$

3.7 Laplace Transform Derivative

Denote $\mathcal{L}\{y(x)\}$ by $Y(s)$. Then under very broad conditions, the Laplace transform of the nth-derivative ($n = 1,2,3,\ldots$) of $y(x)$ is

$$\mathcal{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - s y^{(n-2)}(0) - y^{(n-1)}(0)$$

If the initial conditions on $y(x)$ at $x = 0$ are given by

$$y(0) = c_0, y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1}$$

Note!

Laplace transforms convert differential equations into algebraic equations.

Example 3.4: Solve $y' - 5y = e^{5x};\ y(0) = 0.$

Taking the Laplace transform of both sides of this differential equation and using Property 8.1, we find that $\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{e^{5x}\}$. Then, using the Appendix and Equation 9.4 with $c_0 = 0$, we obtain

$$[sY(s) - 0] - 5Y(s) = \frac{1}{s-5} \text{ from which } Y(s) = \frac{1}{(s-5)^2}$$

Finally, taking the inverse Laplace transform of $Y(s)$, we obtain

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2}\right\} = xe^{5x}$$

5. Post- Test

1. Solve the system.

$$\begin{aligned} y'' + z + y &= 0 \\ z' + y' &= 0; \\ y(0) = 0, \quad y'(0) &= 0, \quad z(0) = 1 \end{aligned}$$

2. **Transfer function of a system is used to calculate which of the following ?**

(a) The order of the system

(b) The time constant

(c) The output for any given input

(d) The steady state gain

Answer Keys

Pre- Test

1. Mathematical Models. Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations.
2. C.2(s+1)/(s+2)

explanation: -c(t)=1-e-2t r(s)=1/s-1/s+1

Self-Test

1.

$$F(s)=\int_0^{\infty} e^{-sx}e^{ax}dx=\lim_{R\to\infty}\int_0^{R} e^{(a-s)x}dx$$

$$=\lim_{R\to\infty}\left[\frac{e^{(a-s)x}}{a-s}\right]_{x=0}^{x=R}=\lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right]$$

$$=\frac{1}{s-a}\,(\text{for } s>a)$$

2.

To the linear factors $s-2$ and $s+1$, we associate respectively the fractions $A/(s-2)$ and $B/(s+1)$. We set

$$\frac{s+3}{(s-2)(s+1)}\equiv\frac{A}{s-2}+\frac{B}{s+1}$$

and, upon clearing fractions, obtain

$$s+3\equiv A(s+1)+B(s-2)$$

To find A and B, we substitute $s=-1$ and $s=2$ into 8.11, we immediately obtain $A=5/3$ and $B=-2/3$. Thus,

$$\frac{s+3}{(s-2)(s+1)}\equiv\frac{5/3}{s-2}-\frac{2/3}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s-2)(s+1)}\right\}=\frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$=\frac{5}{3}e^{2x}-\frac{2}{3}e^{-x}$$

Post- Test

1.

Denote $\mathcal{L}\{y(x)\}$ and $\mathcal{L}\{z(x)\}$ by $Y(s)$ and $Z(s)$ respectively. Then, taking the Laplace transforms of both differential equations, we obtain

$$[s^2Y(s)-(0)s-(0)]+Z(s)+Y(s)=0$$
$$[sZ(s)-1]+[sY(s)-0]=0$$

or

$$(s^2+1)Y(s)+Z(s)=0$$
$$Y(s)+Z(s)=\frac{1}{s}$$

Solving this last system for $Y(s)$ and $Z(s)$, we find that

$$Y(s)=-\frac{1}{s^3} \qquad Z(s)=\frac{1}{s}+\frac{1}{s^3}$$

Thus, taking inverse transforms, we conclude that

$$y(x)=-\frac{1}{2}x^2 \qquad z(x)=1+\frac{1}{2}x^2$$

2. Ans. C

Ministry of high Education and Scientific Research

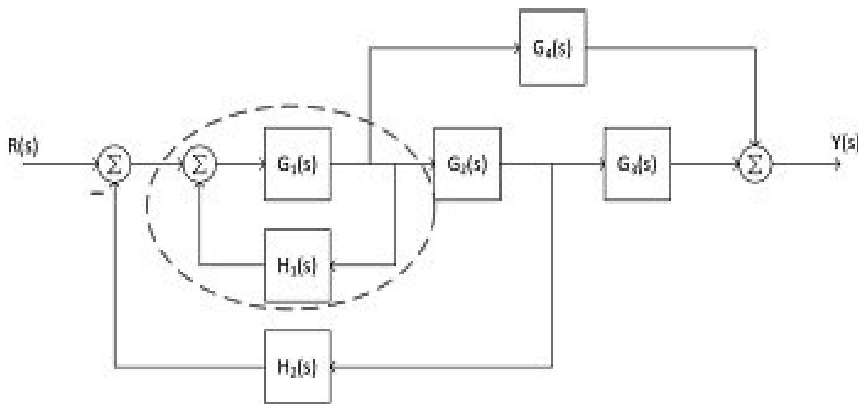
Middle Technical University

Electrical Engineering Technical College

Medical Instrumentation Engineering Techniques Department

- ***Lecture 3 (4^{th}, 5^{th} and 6th Week)***

Transfer function, block diagram representation and reduction, signal flow diagram



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** A Control Systems Engineer is responsible for designing, developing, and implementing solutions that control dynamic systems.

c. **Central Ideas:** Control Systems Engineering is the engineering approach taken to understand how the process can be managed by automation devices and to implement such into operation.

d. **Objectives:** After completing this lecture, the student will be able to:

1. Define a block diagram and describe some applications.
2. It represents the structure of a control system.
3. It helps to organize the variables and equations representing the control system.
4. find a transfer function of a linear system show how some linear systems may be combined together by combining appropriate transfer functions
5. obtain the impulse response and the general response to a linear engineering system

2. Pre-Test:

1. When deriving the transfer function of a linear element?

   A. both initial conditions and loading are taken into account

   B. initial conditions are taken into account but the element is assumed to be not loaded

   C. initial conditions are assumed to be zero but loading is taken into account

   D. initial conditions are assumed to be zero and the element is assumed to be not loaded.

2. signal flow graph is the graphical representation of the relationships between the variables of set linear algebraic equations.

   A. True B. False

Note: Check your answers in "Answer Keys" in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

1. **Theory:**

3.1 Introduction:

In this Section we introduce the concept of a transfer function and then use this to obtain a Laplace transform model of a linear engineering system. (A linear engineering system is one modelled by a constant coefficient ordinary differential equation.)

Linear engineering systems are those that can be modelled by linear differential equations. We shall only consider those systems that can be modelled by constant coefficient ordinary differential equations. Consider a system modelled by the second order differential equation.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

in which a, b, c are given constants and $f(t)$ is a given function. In this context $f(t)$ is often called the **input signal** or **forcing function** and the solution $y(t)$ is often called the **output signal**.
We shall assume that the initial conditions are zero (in this case $y(0) = 0$, $y'(0) = 0$).
Now, taking the Laplace transform of the differential equation, gives:

$$(as^2 + bs + c)Y(s) = F(s)$$

in which we have used $y(0) = y'(0) = 0$ and where we have designated $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{f(t)\} = F(s)$.
We define the **transfer function** of a system to be the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal with the initial conditions as zero. The transfer function (a function of s), is denoted by $H(s)$. In this case

$$H(s) \equiv \frac{Y(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

Now, in the *special case* in which the input signal is the delta function, $f(t) = \delta(t)$, we have $F(s) = 1$ and so,

$$H(s) = Y(s)$$

We call the solution to the differential equation *in this special case* the **unit impulse response function** and denote it by $h(t)u(t)$ (we include the step function $u(t)$ to emphasize its causality). So

$$h(t)u(t) = \mathcal{L}^{-1}\{H(s)\} \qquad \text{when} \quad f(t) = \delta(t)$$

Now, keeping this in mind and returning to the general case in which the input signal $f(t)$ is not necessarily the impulse function $\delta(t)$, we have:

$$Y(s) = H(s)F(s)$$

3.2. Modelling linear systems by transfer functions

We have seen previously that an engineering system can be modelled by one or more differential equations. However, with the introduction of the transfer function we have an alternative model which we examine in this Section.

It will be helpful to develop a pictorial approach to system modelling. To begin, we can imagine a differential equation:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

The system is characterized by the values of the coefficients *a, b, c*. A different engineering system will be characterized by a different set of coefficients. These coefficients are *independent* of the input signal. Changing the input signal does not change the system. It is the system that changes the input signal into the output signal. This is easy to describe pictorially (Figure 3.1).

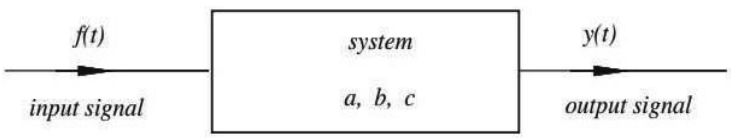


Fig. 3.1: Block diagram describing the system in the t-domain

After the Laplace transform of the differential equation is taken, the differential equation is transformed into in which H(s) is the transfer function. The latter characterizes (in Laplace transform terms) the engineering system from which it was derived. The relation, connecting the Laplace transform of the output Y (s) to the Laplace transform of the input F(s), can also be described schematically (Figure3.2).

$$Y(s) \equiv H(s)F(s) \qquad H(s) \equiv \frac{1}{as^2 + bs + c}$$

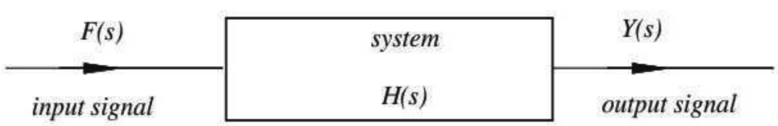


Fig.3.2: Block diagram describing the system in the s-domain

An example of a block diagram is the so-called negative feedback loop, shown in Figure 3.3 (we are using G(s) to denote the transfer function):

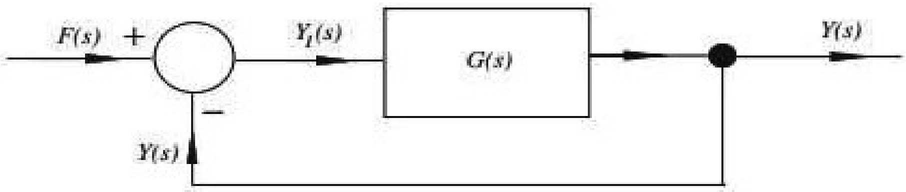


Fig. 3.3: Negative feedback loop

Here, the output signal is tapped and subtracted from the input signal. Hence

$$Y(s) = G(s)Y_1(s)$$

because $Y_1(s)$ is the input signal to the system characterised by transfer function $G(s)$. However, at the summing point $Y_1(s) = F(s) - Y(s)$ and so

$$Y(s) = G(s)(F(s) - Y(s))$$

from which we easily obtain:

$$Y(s) = \left[\frac{G(s)}{1+G(s)}\right] F(s)$$

so that, in terms of input and output signals, the feedback loop is characterised by a transfer function

$$\frac{G(s)}{1+G(s)}.$$

3.3 Laplace Transforms and Impedance

Remember phasor analysis is only valid for sinusoidal steady-state. Turns ac analysis into an analysis similar to the dc. (Ohm's law)

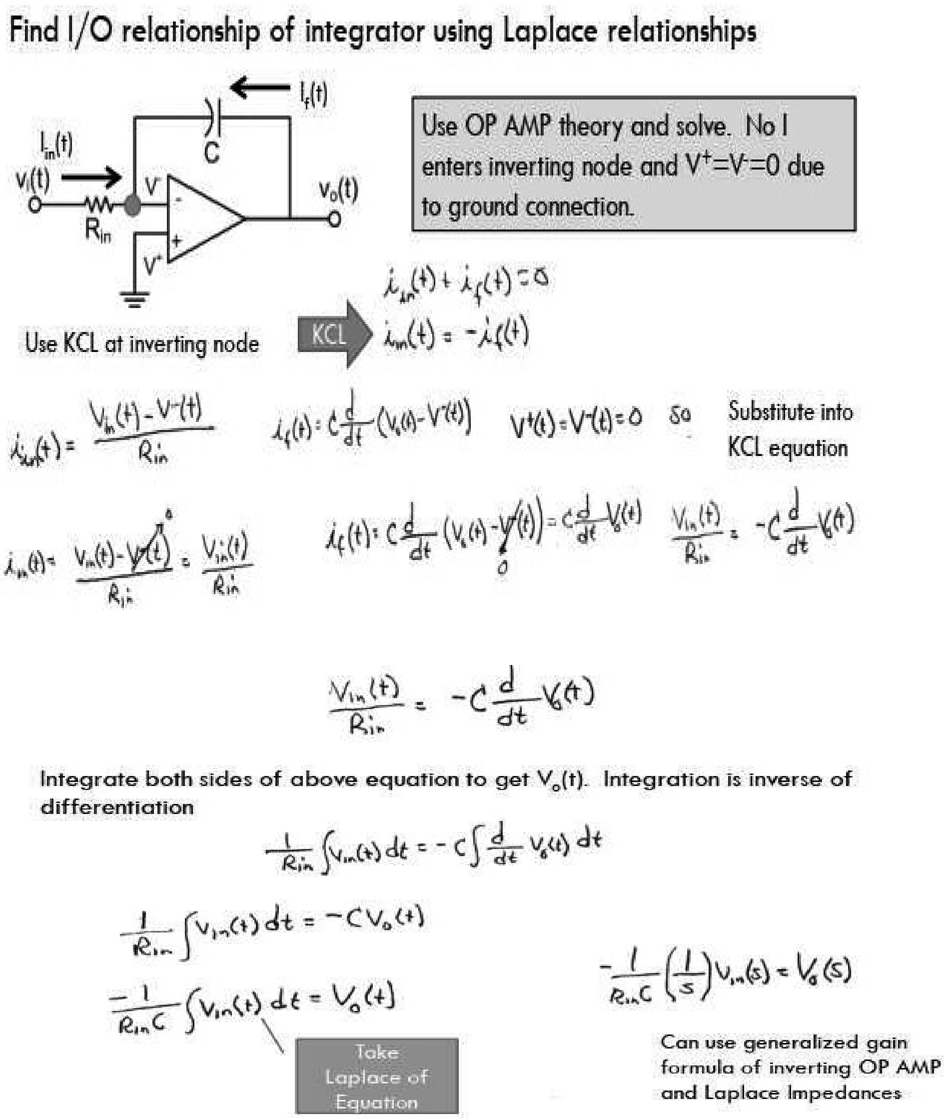
Resistance	R
Inductive Reactance	$X_L = j\cdot\omega\cdot L \quad \omega = 2\pi\cdot f \quad j = 90^\circ$
Capacitive Reactance	$X_C = \frac{1}{j\cdot\omega\cdot C} = -j\cdot\left(\frac{1}{\omega\cdot C}\right) \quad -j = \frac{1}{j} = -90^\circ$

Inductors $Ls = \frac{V_L(s)}{I_L(s)}$ | Inductors $j\omega L = \frac{V_L(j\omega)}{I_L(j\omega)}$

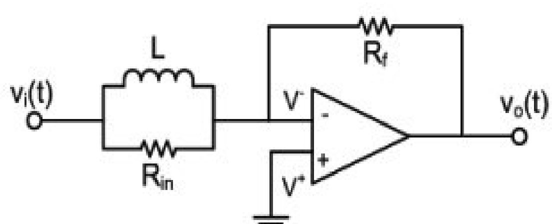
Capacitors $\frac{1}{Cs} = \frac{V_C(s)}{I_C(s)}$ | Capacitors $\frac{1}{j\omega C} = \frac{V_C(j\omega)}{I_C(j\omega)}$

Resistors $R = \frac{V_R(s)}{I_R(s)}$ | Resistors $R = \frac{V_R(j\omega)}{I_R(j\omega)}$

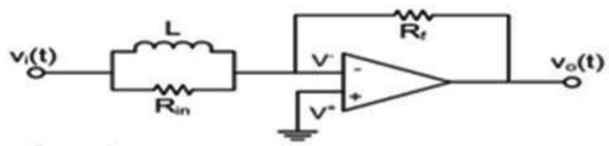
3.4 Laplace Representations of OP AMP Circuits



Example 3.5: Find the input/output relationship for the circuit shown below?



Sol.



Generalized gain formula

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}}$$

Use Laplace impedance relationships to find gain

$$\frac{V_o(s)}{V_{in}(s)} = \frac{-Z_f(s)}{Z_{in}(s)}$$

For inductor

$$V_L(t) = L\frac{d\,i_L(t)}{dt}$$

$$\mathcal{L}[V_L(t)] = V_L(s) \qquad \frac{V_L(s)}{I_L(s)} = Ls$$

$$\mathcal{L}\left[L\frac{d\,i_L(t)}{dt}\right] = Ls\,I_L(s)$$

so $Z_{in}(s) = R_{in} \,||\, Ls$

$$Z_{in}(s) = \frac{R_{in}(Ls)}{R_{in} + Ls}$$

$$Z_f(s) = R_f$$

$$\frac{V_0(s)}{V_{in}(s)} = \frac{-R_f}{\dfrac{R_{in}Ls}{R_{in}+Ls}} = -R_f\left[\frac{R_{in}+Ls}{R_{in}Ls}\right] = \frac{-R_f}{R_{in}}\left[\frac{R_{in}+Ls}{Ls}\right]$$

$$\frac{V_0(s)}{V_{in}(s)} = \frac{-R_f}{R_{in}}\left[\frac{R_{in}}{Ls}+1\right] = \left(\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right)\left(\frac{-R_f}{R_{in}}\right) \qquad \frac{R_{in}}{L} = t\ c$$

$$V_0(s) = \left(\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right)\left(\frac{-R_f}{R_{in}}\right)V_{in}(s)$$

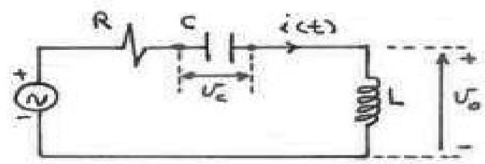
$$-\left(\frac{R_f}{R_{in}}\right)\left(\frac{R_{in}}{L}\right)\left(\frac{1}{s}\right)V_{in}(s)$$

OUTPUT IS SUM OF CONSTANT GAIN And integrator action

$$-\frac{R_f}{Ls}V_{in}(s)$$

Division I Integrati

example : Consider the below electrical network, find the transfer function $\frac{V_o}{V_i}$,



Sol

$$v_i = iR + v_c + L\frac{di}{dt}$$

taking Laplace trasform

$$\rightarrow V_i = IR + V_c + LSI \ldots (1)$$, assuming zero initial Conditions

now, $i = C\frac{dv_c}{dt}$

$$\rightarrow I = CSV_c \rightarrow V_c = \frac{I}{CS} \quad \ldots (2)$$

sub. in (1)

$$\rightarrow V_i = IR + \frac{I}{CS} + LSI$$

$$v_o = L\frac{di}{dt} \rightarrow V_o = LSI$$

$$\rightarrow \text{Transfer function} = \frac{V_o}{V_i} = \frac{LSI(s)}{I(s)\left[R+LS+\frac{1}{CS}\right]}$$

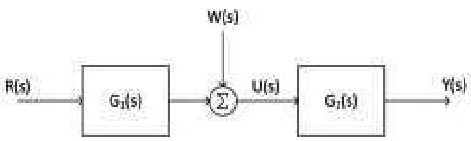
$$\rightarrow \frac{V_o}{V_i} = \frac{S^2}{S^2+\frac{R}{L}S+\frac{1}{LC}}$$

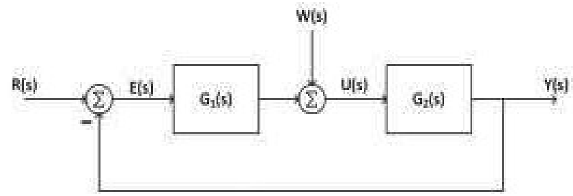
4. Self- Test

1. Find the input/output relationship for the differentiator.?
2. Find T.F. (Eo/Ei) for RC – series circuit when C on the output?

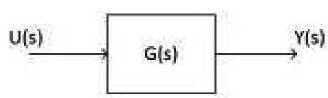
3.5 Block diagram

- In the introductory section we saw examples of ***block diagrams*** to represent systems, e.g.:





- Block diagrams consist of
  - ***Blocks*** – these represent subsystems – typically modeled by, and labeled with, a transfer function
  - ***Signals*** – inputs and outputs of blocks – signal direction indicated by arrows – could be voltage, velocity, force, etc.
  - ***Summing junctions*** – points were signals are algebraically summed – subtraction indicated by a negative sign near where the signal joins the summing junction
- The basic input/output relationship for a single block is:



$$Y(s) = U(s) \cdot G(s)$$

- Block diagram blocks can be connected in three basic forms:
  - ***Cascade***
  - ***Parallel***
  - ***Feedback***
- We'll next look at each of these forms and derive a single-block equivalent for each

- **Cascade Form**

- Blocks connected in ***cascade***:

$$X_1(s) = U(s) \cdot G_1(s), \quad X_2(s) = X_1(s) \cdot G_2(s)$$

$$Y(s) = X_2(s) \cdot G_3(s) = X_1(s) \cdot G_2(s) \cdot G_3(s)$$

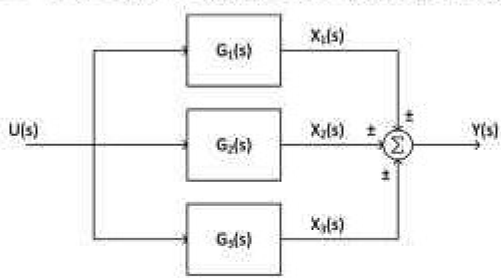
$$Y(s) = U(s) \cdot G_1(s) \cdot G_2(s) \cdot G_3(s) = U(s) \cdot G_{eq}(s)$$

$$G_{eq}(s) = G_1(s) \cdot G_2(s) \cdot G_3(s)$$

- The equivalent transfer function of cascaded blocks is the ***product*** of the individual transfer functions

- **Parallel Form**

- Blocks connected in parallel:



$$X_1(s) = U(s) \cdot G_1(s)$$

$$X_2(s) = U(s) \cdot G_2(s)$$

$$X_3(s) = U(s) \cdot G_3(s)$$

$$Y(s) = X_1(s) \pm X_2(s) \pm X_3(s)$$

$$Y(s) = U(s) \cdot G_1(s) \pm U(s) \cdot G_2(s) \pm U(s) \cdot G_3(s)$$

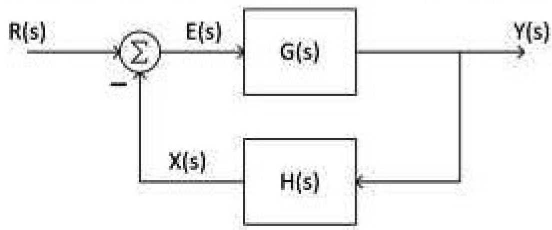
$$Y(s) = U(s)[G_1(s) \pm G_2(s) \pm G_3(s)] = U(s) \cdot G_{eq}(s)$$

$$G_{eq}(s) = G_1(s) \pm G_2(s) \pm G_3(s)$$

- The equivalent transfer function is the ***sum*** of the individual transfer functions:

- **Feedback Form**

□ Of obvious interest to us, is the ***feedback form***:



$$Y(s) = E(s)G(s)$$

$$Y(s) = [R(s) - X(s)]G(s)$$

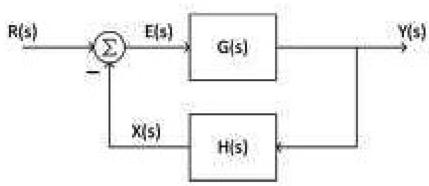
$$Y(s) = [R(s) - Y(s)H(s)]G(s)$$

$$Y(s)[1 + G(s)H(s)] = R(s)G(s)$$

$$Y(s) = R(s) \cdot \frac{G(s)}{1 + G(s)H(s)}$$

□ The ***closed-loop transfer function***, $T(s)$, is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$



$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

□ Note that this is ***negative feedback***, for ***positive feedback***:

$$T(s) = \frac{G(s)}{1 - G(s)H(s)}$$

□ The $G(s)H(s)$ factor in the denominator is the ***loop gain*** or ***open-loop transfer function***

□ The gain from input to output with the feedback path broken is the ***forward path gain*** – here, $G(s)$

□ In general:

$$T(s) = \frac{\text{forward path gain}}{1 - \text{loop gain}}$$

3.6 Block Diagram Algebra

- Often want to simplify block diagrams into simpler, recognizable forms
  - To determine the equivalent transfer function
- Simplify to instances of the three standard forms, then simplify those forms
- ***Move blocks around relative to summing junctions and pickoff points*** – simplify to a standard form
  - Move blocks forward/backward past summing junctions
  - Move blocks forward/backward past pickoff points

- **Note: Obtaining Cascaded, Parallel, and Feedback (Closed-Loop) Transfer Functions with MATLAB**

$$G_1(s) = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{\text{num2}}{\text{den2}}$$

To obtain the transfer functions of the cascaded system, parallel system, or feedback (closed-loop) system, the following commands may be used:

```
[num, den] = series(num1,den1,num2,den2)
[num, den] = parallel(num1,den1,num2,den2)
[num, den] = feedback(num1,den1,num2,den2)
```

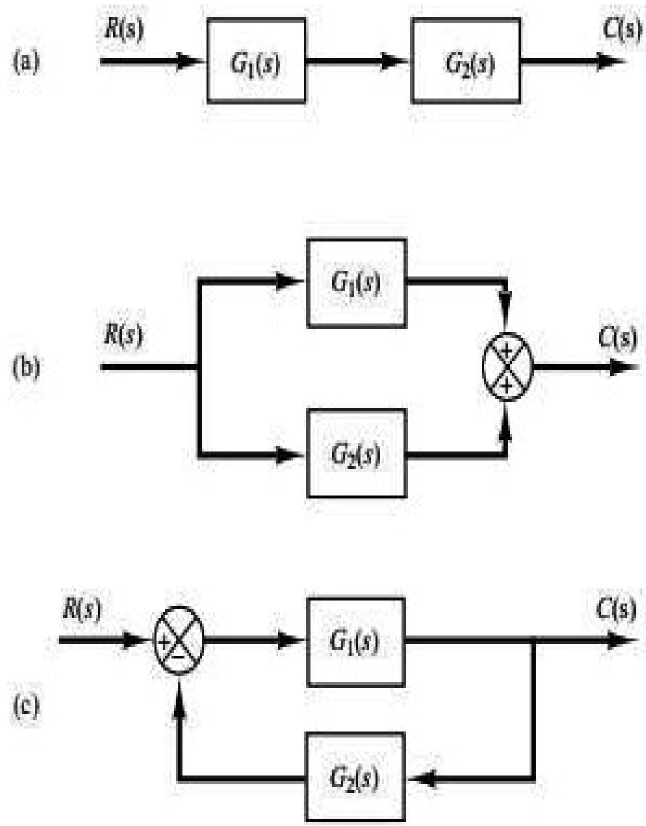
As an example, consider the case where

$$G_1(s) = \frac{10}{s^2 + 2s + 10} = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{5}{s + 5} = \frac{\text{num2}}{\text{den2}}$$

MATLAB Program 2–1 gives $C(s)/R(s) = \text{num}/\text{den}$ for each arrangement of $G_1(s)$ and $G_2(s)$. Note that the command

```
printsys(num,den)
```

displays the num/den [that is, the transfer function $C(s)/R(s)$] of the system considered.



MATLAB Program 2–1

```
num1 = [10];
den1 = [1  2  10];
num2 = [5];
den2 = [1  5];
[num, den] = series(num1,den1,num2,den2);
printsys(num,den)

num/den =

           50
 -----------------------
 s^3 + 7s^2 + 20s + 50

[num, den] = parallel(num1,den1,num2,den2);
printsys(num,den)

num/den =

   5s^2 + 20s + 100
 -----------------------
 s^3 + 7s^2 + 20s + 50

[num, den] = feedback(num1,den1,num2,den2);
printsys(num,den)

num/den =

        10s + 50
 ------------------------
 s^3 + 7s^2 + 20s + 100
```

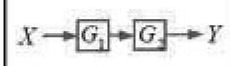
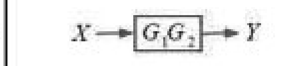
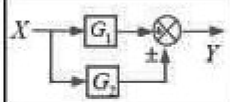
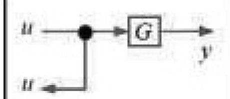
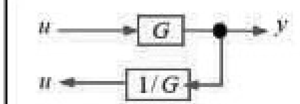
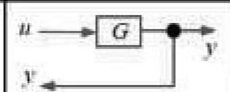
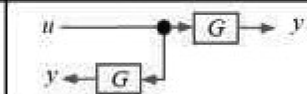
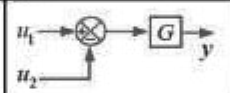
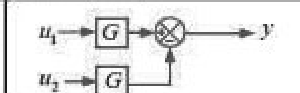
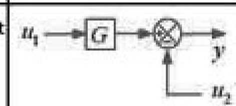
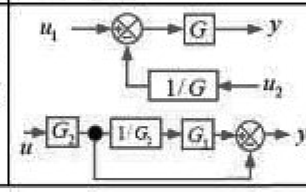
3.7 Block Diagram Reduction

3.7.1 Block Diagram Reduction Rules

Table 1: Block Diagram Reduction Rules

1.	**Combine all cascade blocks**
2.	**Combine all parallel blocks**
3.	**Eliminate all minor (interior) feedback loops**
4.	**Shift summing points to left**
5.	**Shift takeoff points to the right**
6.	**Repeat Steps 1 to 5 until the canonical form is obtained**

Table 2: Basic rules with block diagram transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y=(G_1G_2)X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y=(G_1 \pm G_2)X$
3	Moving a pickoff point behind a block			$y = Gu$ $u = \frac{1}{G}y$
4	Moving a pickoff point ahead of a block			$y = Gu$
5	Moving a summing point behind a block			$e_2 = G(u_1 - u_2)$
6	Moving a summing point ahead of a block			$y = Gu_1 - u_2$ $y=(G_1-G_2)u$

Example 3.6: Simplify the block diagram shown in Figure 3.4 to 3.5, Obtain the transfer function relating C(s) and R(s)?

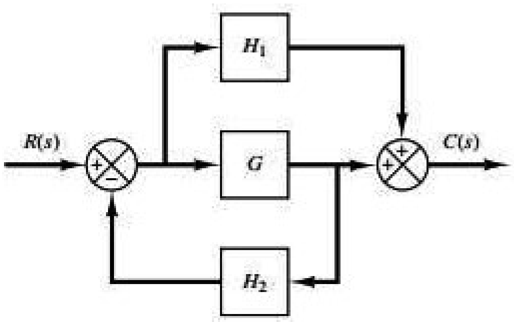


Fig.3.4

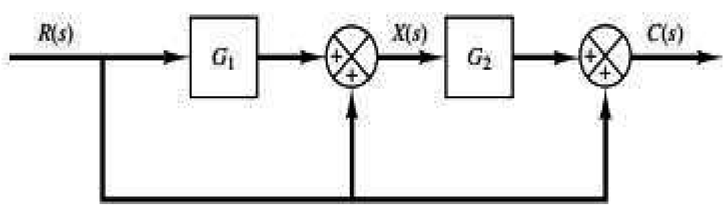


Fig.3.5

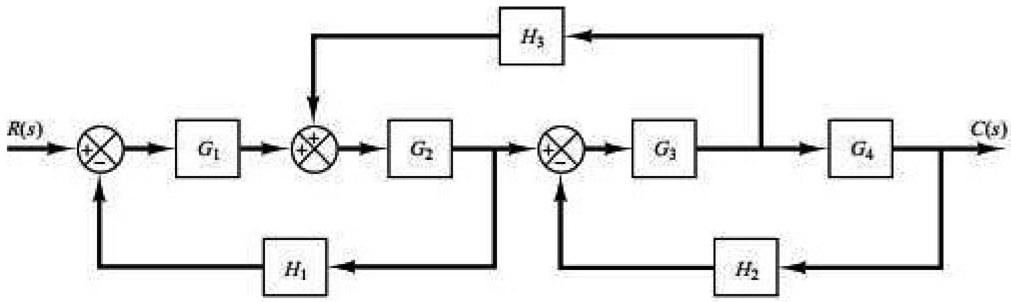
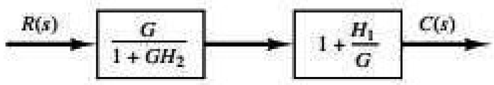


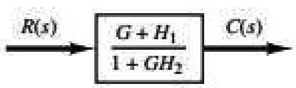
Fig.3.6

Ans. of Fig. 3.4

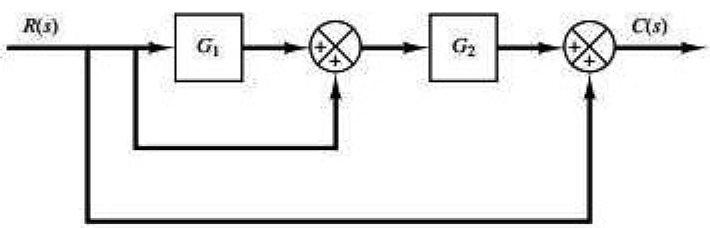
(b)



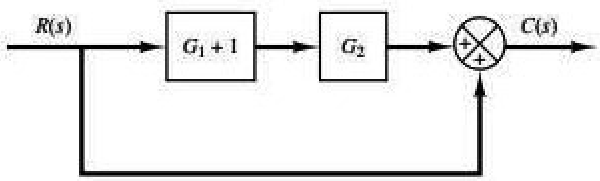
(c)



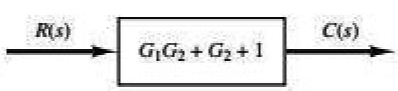
Ans. of Fig. 3.5



(a)



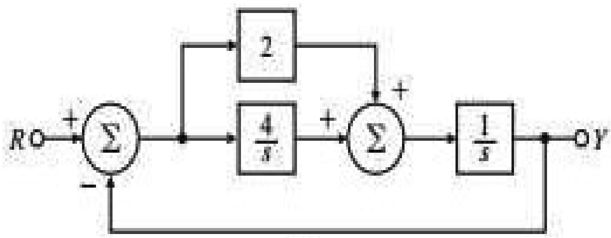
(b)



(c)

3.7.1. Transfer Function of a Simple System Using Matlab

Example: Simplify the block diagram shown in Figure 3.7 Then obtain the closed-loop by using matlab?



(a)

Ans.

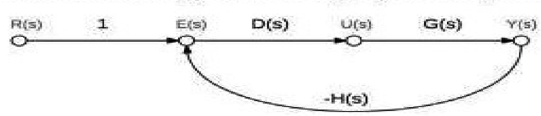
```
s=tf('s');                        % specify a transfer function using
                                     a rational function in the Laplace
                                     variable s
sysG1=2;                          % define subsystem G1
sysG2=4/s;                        % define subsystem G2
sysG3=parallel(sysG1,sysG2);      % parallel combination of G1 and G2
sysG4=1/s;                        % define subsystem G4
sysG5=series(sysG3,sysG4);        % series combination of G3 and G4
sysG6=1;
sysCL=feedback(sysG5,sysG6,-1);   % feedback combination of G5 and G6
```

The Matlab results are sysCL of the form

$$\frac{Y(s)}{R(s)} = \frac{2s+4}{s^2+2s+4},$$

3.8. Mason's Rule and the Signal Flow Graph

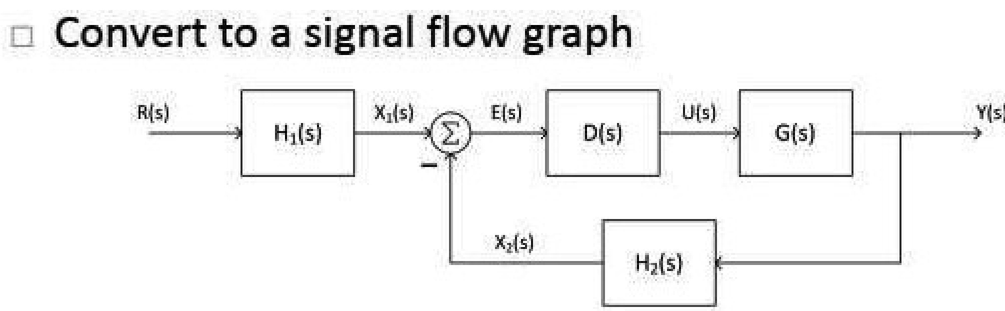
- An alternative to block diagrams for graphically describing systems

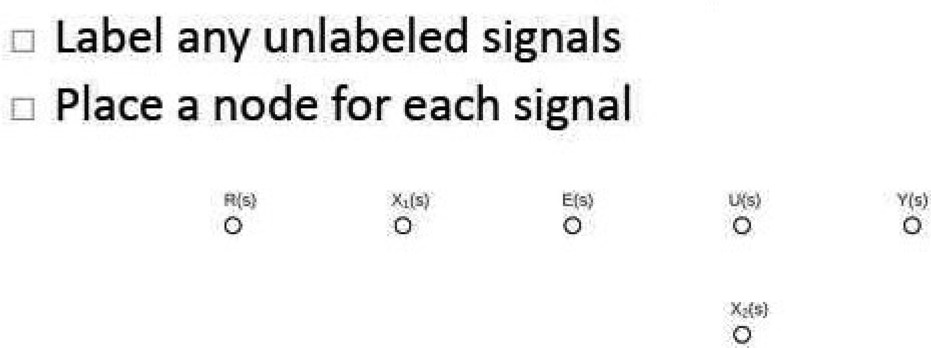


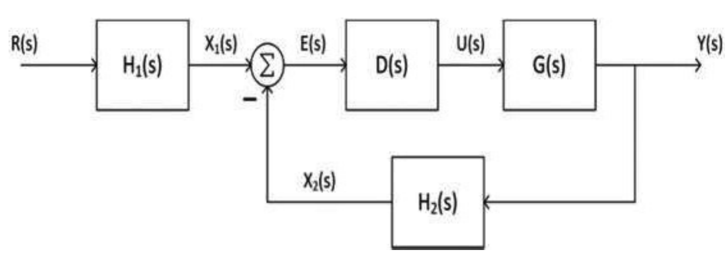
- Signal flow graphs consist of:
  - ***Nodes*** –represent signals
  - ***Branches*** –represent system blocks
- Branches labeled with system transfer functions
- Nodes (sometimes) labeled with signal names
- Arrows indicate signal flow direction
- Implicit summation at nodes
  - Always a positive sum
  - Negative signs associated with branch transfer functions

3.8.1 Block Diagram Signal Flow Graph

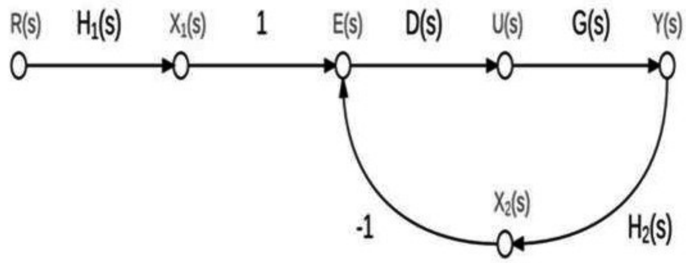
To convert from a block diagram to a signal flow graph:
1. Identify and label all signals on the block diagram
2. Place a node for each signal
3. Connect nodes with branches in place of the blocks
    - Maintain correct direction
    - Label branches with corresponding transfer functions
    - Negate transfer functions as necessary to provide negative feedback
4. If desired, simplify where possible



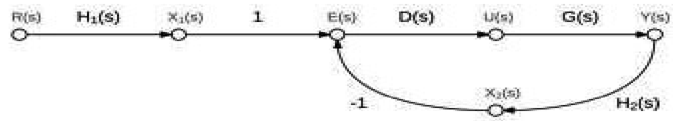




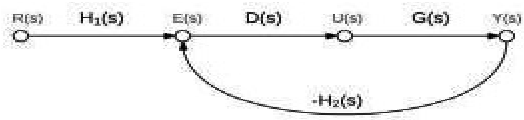
Connect nodes with branches, each representing a system block



Note the -1 to provide negative feedback of $X_1(s)$

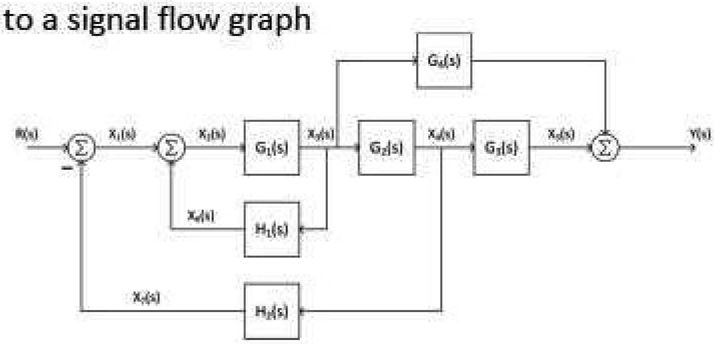


- Nodes with a single input and single output can be eliminated, if desired
  - This makes sense for $X_1(s)$ and $X_2(s)$
  - Leave $U(s)$ to indicate separation between controller and plant

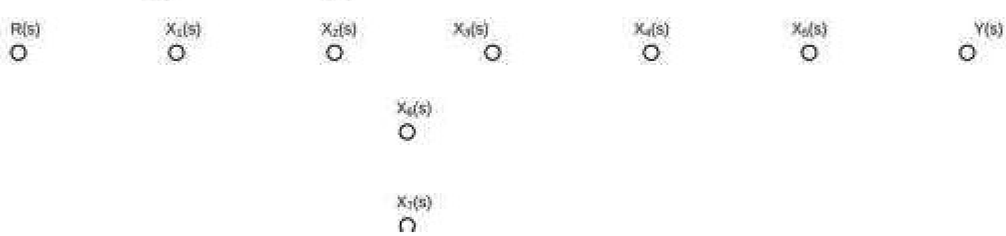


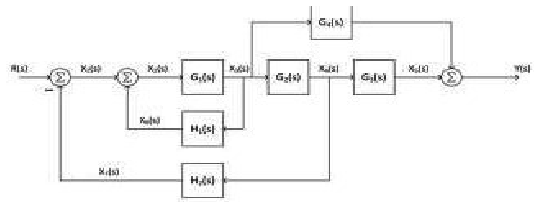
Example:

- Revisit the block diagram from earlier
  - Convert to a signal flow graph

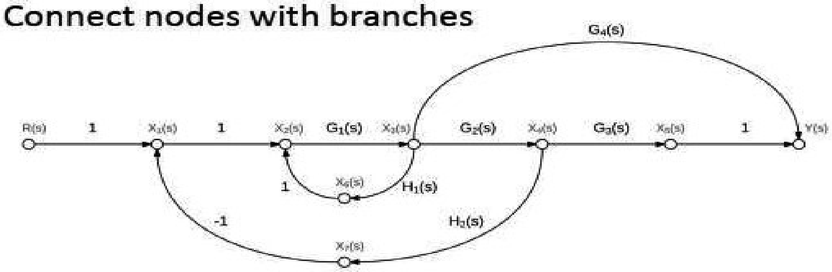


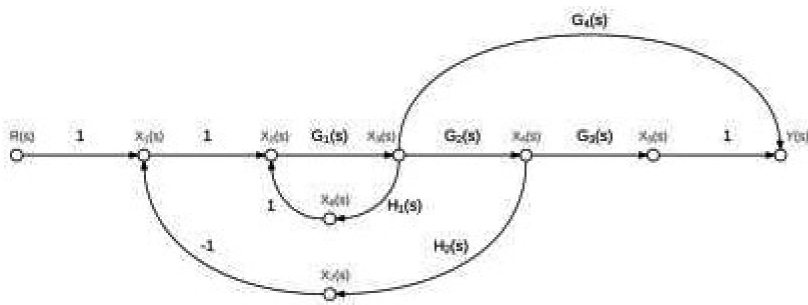
- Label all signals, then place a node for each



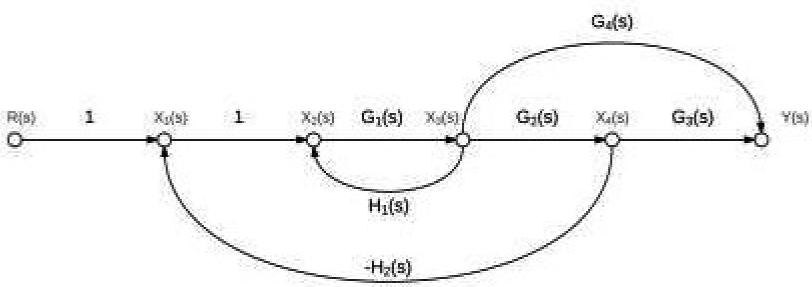


- Connect nodes with branches





- Simplify – eliminate $X_5(s)$, $X_6(s)$, and $X_7(s)$



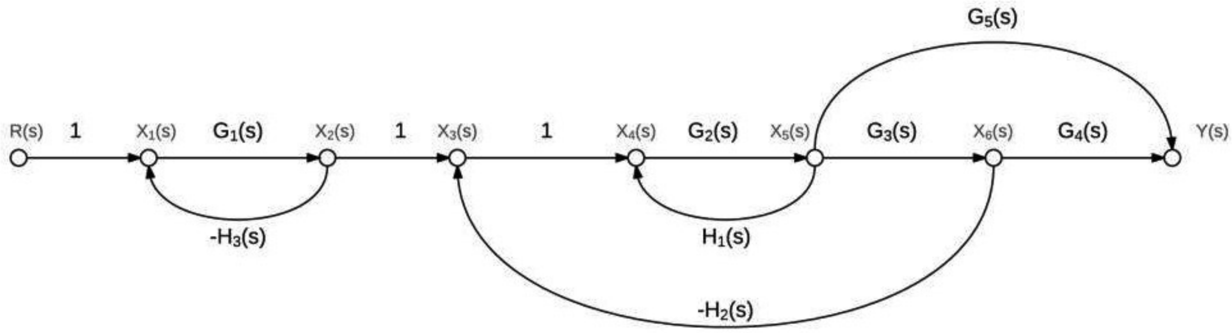
3.9 Signal Flow Graphs vs. Block Diagrams

- Signal flow graphs and block diagrams are alternative, though equivalent, tools for graphical representation of interconnected systems
- A generalization (not a rule)
- Signal flow graphs – more often used when dealing with state-space system models
- Block diagrams – more often used when dealing with transfer function system models

3.10 Mason's Rule

We've seen how to reduce a complicated block diagram to a single input-to-output transfer function

- Many successive simplifications
- Mason's rule provides a formula to calculate the same overall transfer function
- Single application of the formula
- Can get complicated
- Before presenting the Mason's rule formula, we need to define some terminology



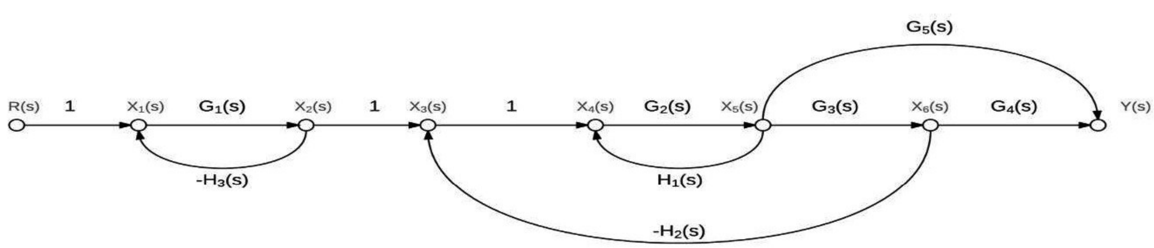
Loop gain – total gain (product of individual gains) around any path in the signal flow graph

- Beginning and ending at the same node
- Not passing through any node more than once
- Here, there are three loops with the following gains:

1. **$-G_1H_3$**
2. **G_2H_1**
3. **$-G_2G_3H_2$**

Forward Path Gain

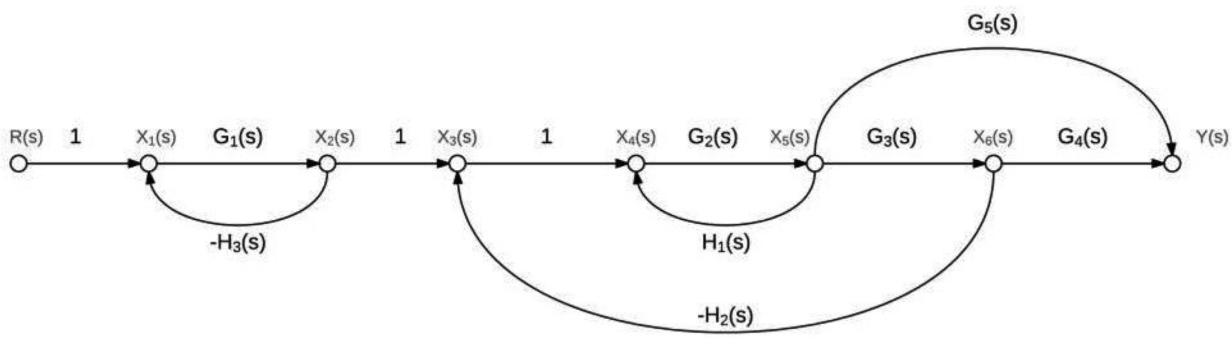
- ***Forward path gain*** – gain along any path from the input to the output
  - Not passing through any node more than once
- Here, there are two forward paths with the following gains:
  1. $G_1G_2G_3G_4$
  2. $G_1G_2G_5$



Non-Touching Loops

- ***Non-touching loops*** – loops that do not have any nodes in common
- Here,
  1. $-G_1H_3$ does not touch G_2H_1
  2. $-G_1H_3$ does not touch $-G_2G_3H_2$

Non-Touching Loop Gains



- ***Non-touching loop gains*** – the *product* of loop gains from non-touching loops, taken two, three, four, or more at a time
- Here, there are only two *pairs* of non-touching loops
  1. $[-G_1H_3] \cdot [G_2H_1]$
  2. $[-G_1H_3] \cdot [-G_2G_3H_2]$

SUMMARY OF MASON'S RULE

$$T(s) = \frac{Y(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{P} T_k \Delta_k$$

where

P = # of forward paths

T_k = gain of the k^{th} forward path

$\Delta = 1 - \Sigma$(loop gains)

$+\Sigma$(non-touching loop gains taken two-at-a-time)

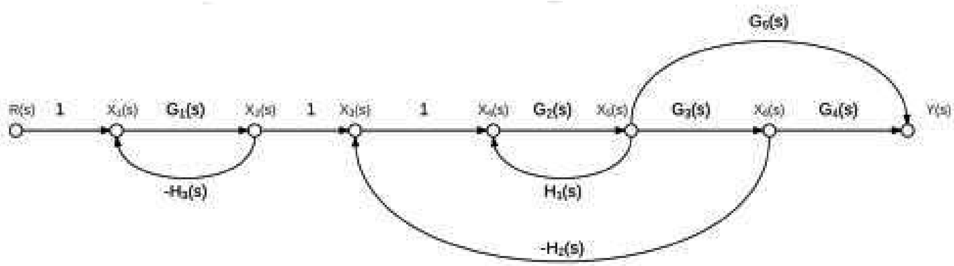
$-\Sigma$(non-touching loop gains taken three-at-a-time)

$+\Sigma$(non-touching loop gains taken four-at-a-time)

$-\Sigma$ ...

$\Delta_k = \Delta - \Sigma$(loop gain terms in Δ that touch the k^{th} forward path)

EXAMPLE:



- # of forward paths:

  $P = 2$

- Forward path gains:

  $T_1 = G_1G_2G_3G_4$

  $T_2 = G_1G_2G_5$

- Σ(loop gains):

  $-G_1H_3 + G_2H_1 - G_2G_3H_2$

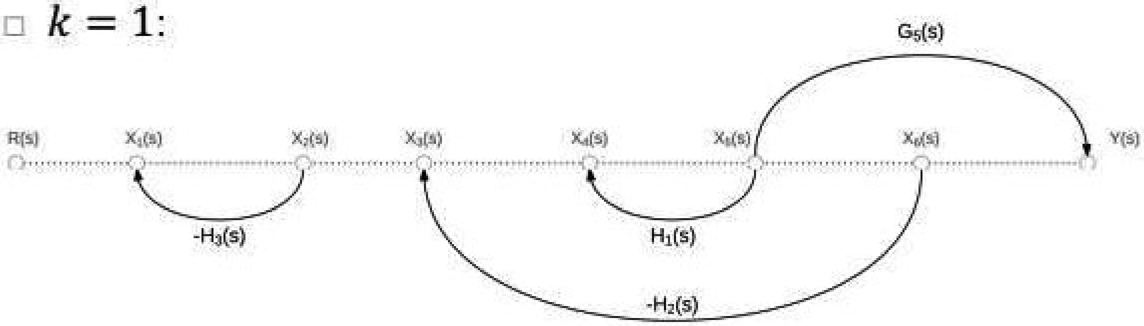
- Σ(NTLGs taken two-at-a-time):

  $(-G_1H_3G_2H_1) + (G_1H_3G_2G_3H_2)$

- Δ:

  $$\Delta = 1 - (-G_1H_3 + G_2H_1 - G_2G_3H_2) + (-G_1H_3G_2H_1 + G_1H_3G_2G_3H_2)$$

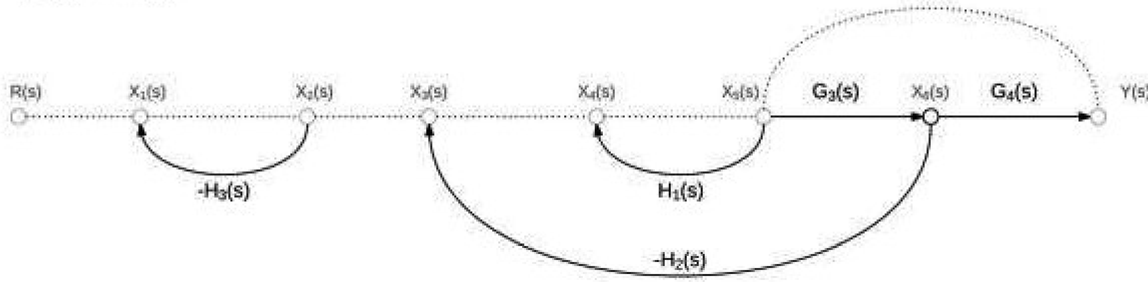
- Simplest way to find Δ_k terms is to calculate Δ with the k^{th} path removed – must remove *nodes* as well
- $k = 1$:



- With forward path 1 removed, there are no loops, so

$$\Delta_1 = 1 - 0$$
$$\Delta_1 = 1$$

- $k = 2$:



- Similarly, removing forward path 2 leaves no loops, so

$$\Delta_2 = 1 - 0$$
$$\Delta_2 = 1$$

- For our example:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{P} T_k \Delta_k$$

$P = 2$

$T_1 = G_1 G_2 G_3 G_4$

$T_2 = G_1 G_2 G_5$

$\Delta = 1 + G_1 H_3 - G_2 H_1 + G_2 G_3 H_2 - G_1 H_3 G_2 H_1 + G_1 H_3 G_2 G_3 H_2$

$\Delta_1 = 1$

$\Delta_2 = 1$

- The closed-loop transfer function:

$$T(s) = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

$$T(s) = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_5}{1 + G_1 H_3 - G_2 H_1 + G_2 G_3 H_2 - G_1 H_3 G_2 H_1 + G_1 H_3 G_2 G_3 H_2}$$

5. Post- Test

1. Simplify the block diagram shown in Figure 3.6 Then obtain the closed-loop transfer function C(s)/R(s).

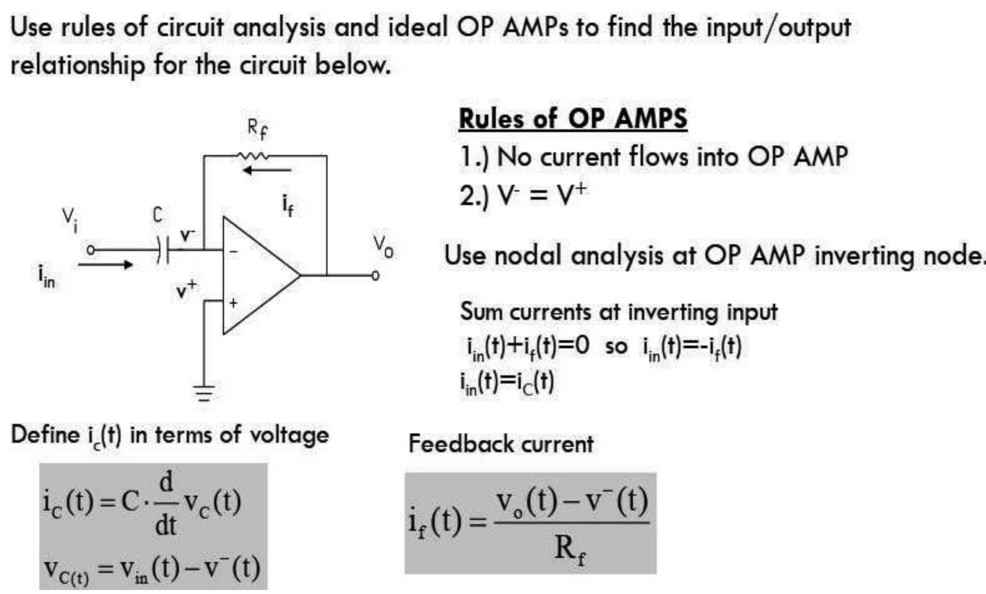
Answer Keys

Pre- Test

1. C- initial conditions are assumed to be zero but loading is taken into account
   explanation: -when deriving the transfer function of a linear element only initial conditions are assumed to be zero, loading cannot be assumed to be zero
2. True.

Self-Test

1.



2.

Complete derivation

$$i_{in}(t) = -i_f(t)$$

$$C \cdot \frac{d}{dt}\left[v_{in}(t) - v^-(t)\right] = -\left[\frac{v_o(t) - v^-(t)}{R_f}\right]$$

$V^+(t)=V^-(t)=0$ Positive terminal grounded

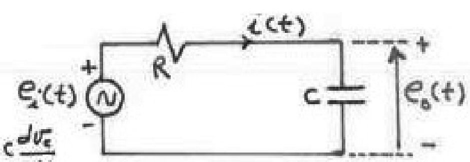
$$C \cdot \frac{d}{dt}\left[v_{in}(t)\right] = \left[\frac{-v_o(t)}{R_f}\right]$$

$$-R_f \cdot C \cdot \frac{d}{dt}\left[v_{in}(t)\right] = v_o(t)$$

Input

Circuit takes derivative of input voltage

Output



sol

$e_i(t) = i(t)R + V_c(t)$; and $i(t) = C\frac{dV_c}{dt}$

$\rightarrow E_i(s) = I(s)R + \frac{I(s)}{Cs}$ ....(1)

and $E_o = V_c(s) = \frac{I(s)}{Cs}$ ---(2)

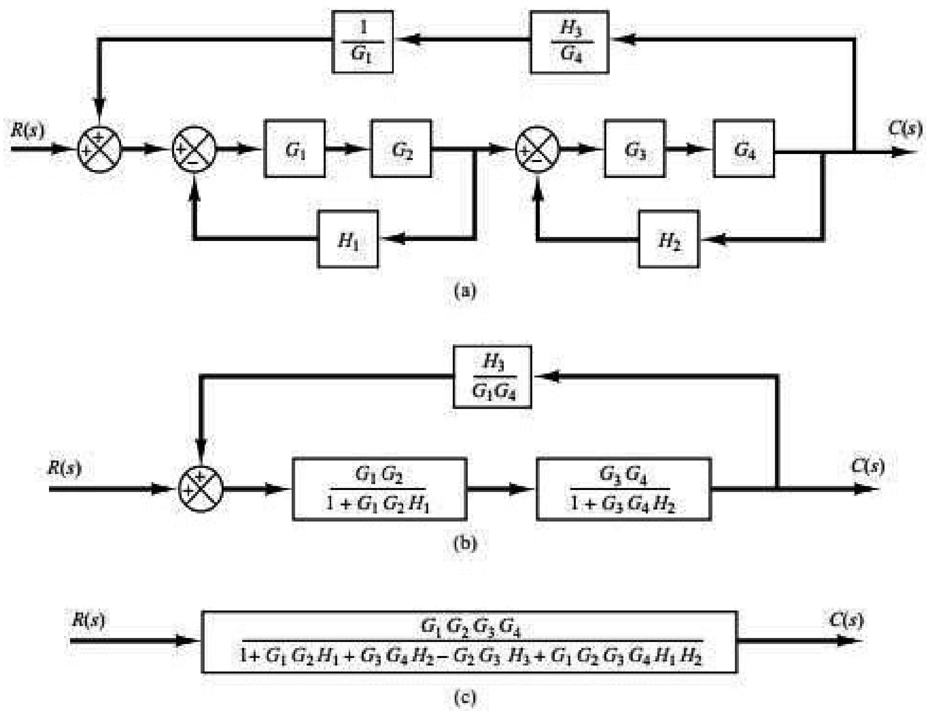
$\rightarrow$ Transfer function $= \frac{E_o(s)}{E_i(s)} = \frac{\frac{I(s)}{Cs}}{I(s)R + \frac{I(s)}{Cs}}$

$= \frac{1}{RCs+1} = \frac{1}{Ts+1}$

where $T = RC$

Post- Test

1.



(a)
(b)
(c)
R(s)
C(s)
G1
G2
G3
G4
H1
H2
1/G1
H3/G4
H3/(G1G4)
G1G2/(1 + G1G2H1)
G3G4/(1 + G3G4H2)
G1G2G3G4/(1 + G1G2H1 + G3G4H2 − G2G3H3 + G1G2G3G4H1H2)

Ministry of high Education and Scientific Research

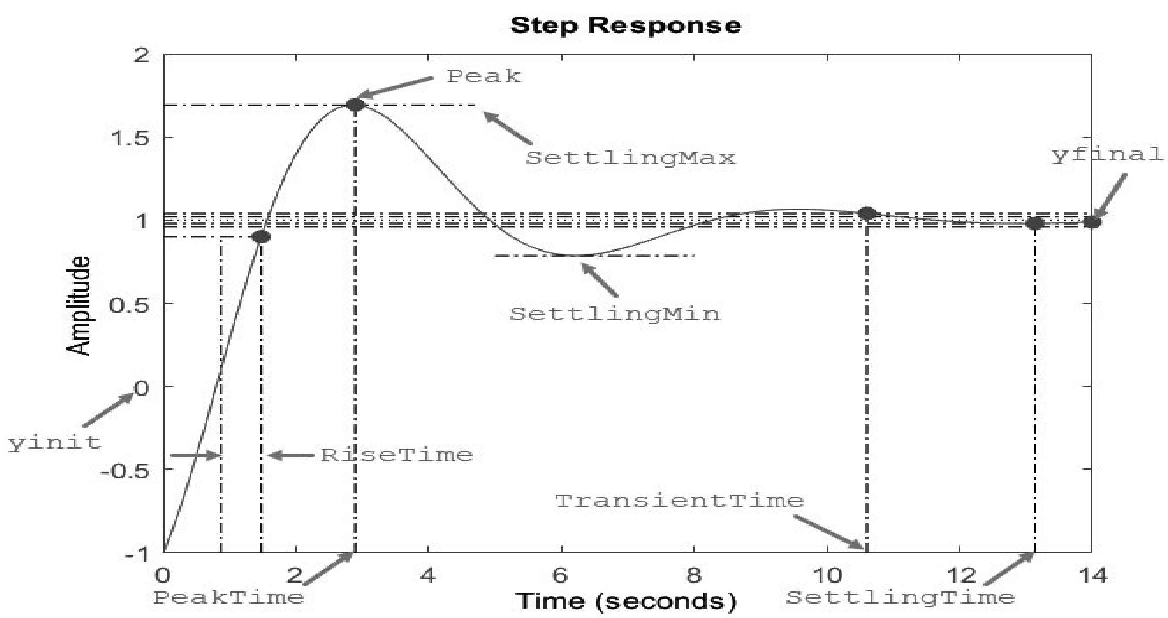
Middle Technical University

Electrical Engineering Technical College

Medical Instrumentation Engineering Techniques Department

- ***Lecture 4 (7^{th} , 8^{th} , 9^{th} Week)***

Time domain analysis, steady – state transient analysis



For

Students of Fourth Stage

Medical Instrument Department

By

Asst. prof. Dr. Ahmed R. Ajel

Department of Medical Instrumentation Engineering Techniques

1. Overview

a. **Target Population:** For students of fourth stage for Medical Instrument Department in Electrical Engineering Technical College

b. **Rationale:** A Control Systems Engineer is responsible for designing, developing, and implementing solutions that control dynamic systems.

c. **Central Ideas:** Control Systems Engineering is the engineering approach taken to understand how the process can be managed by automation devices and to implement such into operation.

d. **Objectives:** After completing this lecture, the student will be able to:

1. Define time response specification.

2. Describe steady state error

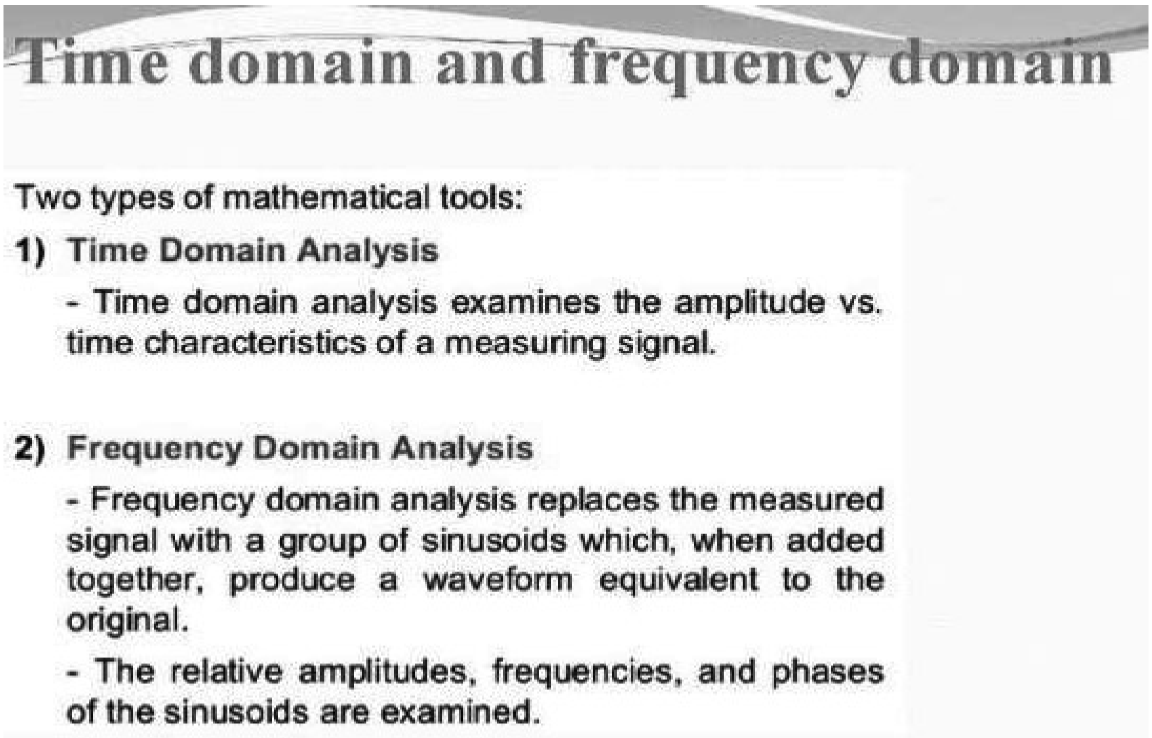
2. Pre-Test:

1. The time response of linear system is the addition of transient response which depend on preliminary conditions. State True or false
2. the steady-state response which is based on output of system? State True or false.

Note: Check your answers in "Answer Keys" in end of unit. If you obtain 75% of solution, you cannot need to this unit. If your answer is poor, you will transfer to next page.

3. Theory:

3.1 Introduction:



3.2 Typical Test Signals

The commonly used test input signals are step functions, ramp functions, acceleration functions, impulse functions, sinusoidal functions, and white noise. In this chapter we use test signals such as step, ramp, acceleration and impulse signals. With these test signals, mathematical and experimental analyses of control systems can be carried out easily, since the signals are very simple functions of time Which of these typical input signals to use for analyzing system characteristics may be determined by the form of the input that the system will be subjected to most frequently under normal operation. If the inputs to a control system are gradually changing functions of time, then a ramp function of time may be a good test signal. Similarly, if a system is subjected to sudden disturbances, a step function of time may be a good test signal; and for a system subjected to shock inputs, an impulse function may be best. Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the performance of many systems on the same basis.

3.3 **Transient Response and Steady-State Response**

The time response of a control system consists of two parts: the transient response and the steady-state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus the system response c(t) may be written as where the first term on the right-hand side of the equation is the transient response and the second term is the steady-state response.

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

where the first term on the right-hand side of the equation is the transient response and

the second term is the steady-state response.

3.4 First-Order Systems

Consider the first-order system shown in Figure 5–1(a). Physically, this system may represent an RC circuit, thermal system, or the like. A simplified block diagram is shown in Figure 5–1(b). The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp, and unit-impulse functions. The initial conditions are assumed to be zero.

Note that all systems having the same transfer function will exhibit the same output in response to the same input. For any given physical system, the mathematical response can be given a physical interpretation.

Unit-Step Response of First-Order Systems. Since the Laplace transform of the unit-step function is $1/s$, substituting $R(s) = 1/s$ into Equation (5–1), we obtain

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)}$$

Taking the inverse Laplace transform of Equation (5–2), we obtain

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

Equation (5–3) states that initially the output $c(t)$ is zero and finally it becomes unity. One important characteristic of such an exponential response curve $c(t)$ is that at $t = T$ the value of $c(t)$ is 0.632, or the response $c(t)$ has reached 63.2% of its total change. This may be easily seen by substituting $t = T$ in $c(t)$. That is,

$$c(T) = 1 - e^{-1} = 0.632$$

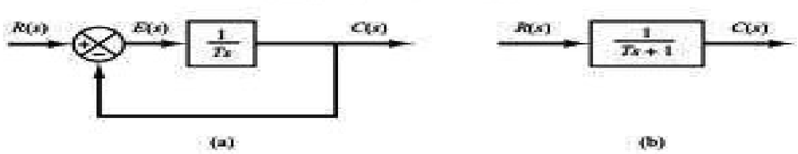


FIG. 1: (a) Block diagram of a first-order system; (b) simplified block diagram

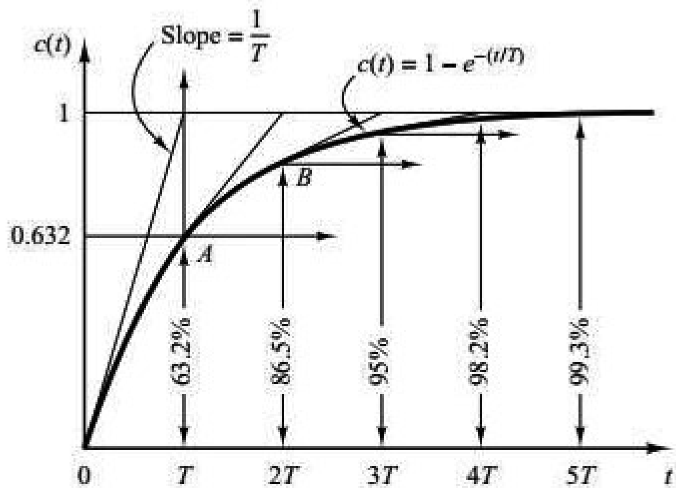


FIG.: Exponential response curve.

Unit-Ramp Response of First-Order Systems. Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of Figure 5–1(a) as

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s^2}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1}$$

Taking the inverse Laplace transform of Equation (5–5), we obtain

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

The error signal $e(t)$ is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T\left(1 - e^{-t/T}\right) \end{aligned}$$

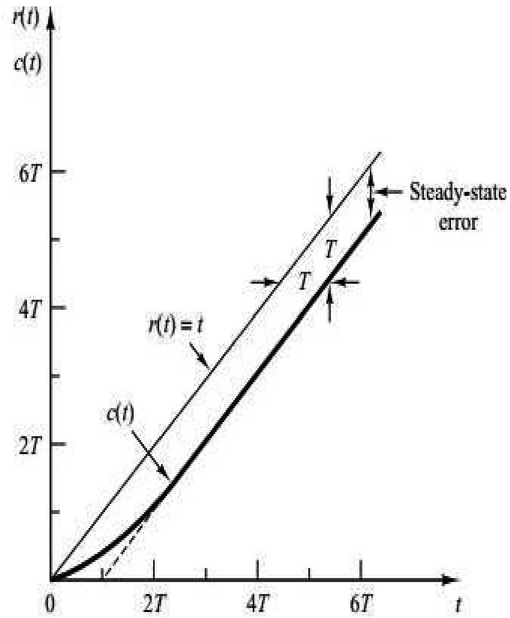


FIG: Unit-ramp response of the system.

4. Self- Test

1. Draw Unit-Impulse Response of First-Order Systems?

3.5 Second-Order Systems

The closed-loop transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

Step Response of Second-Order System. The closed-loop transfer function of the system shown in Figure 5–5(c) is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} \tag{5–9}$$

which can be rewritten as

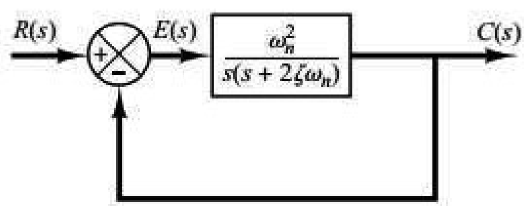
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2 - 4JK < 0$ and they are real if $B^2 - 4JK \geq 0$. In the transient-response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2, \qquad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where σ is called the *attenuation*; ω_n, the *undamped natural frequency*; and ζ, the *damping ratio* of the system. The damping ratio ζ is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{JK}$ or

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$



In terms of ζ and ω_n, the system shown in Figure 5–5(c) can be modified to that shown in Figure 5–6, and the closed-loop transfer function $C(s)/R(s)$ given by Equation can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(2) *Critically damped case* ($\zeta = 1$): If the two poles of $C(s)/R(s)$ are equal, the system is said to be a critically damped one.

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

The inverse Laplace transform of Equation (5–14) may be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0$$

This result can also be obtained by letting ζ approach unity in Equation (5–12) and by using the following limit:

$$\lim_{\zeta \to 1} \frac{\sin \omega_d t}{\sqrt{1 - \zeta^2}} = \lim_{\zeta \to 1} \frac{\sin \omega_n \sqrt{1 - \zeta^2}\, t}{\sqrt{1 - \zeta^2}} = \omega_n t$$

(3) *Overdamped case* ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The inverse Laplace transform of Equation (5–16) is

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0 \end{aligned}$$

where $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$. Thus, the response $c(t)$ includes two decaying exponential terms.

Definitions of Transient-Response Specifications. Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input, since it is easy to generate and is sufficiently drastic. (If the response to a step input is known, it is mathematically possible to compute the response to any input.) The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, t_d

2. Rise time, t_r

3. Peak time, t_p

4. Maximum overshoot, M_p

5. Settling time, t_s

Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by:

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100'$$

The time-domain specifications just given are quite important, since most control systems are time-domain systems; that is, they must exhibit acceptable time responses. (This means that, the control system must be modified until the transient response is satisfactory.)

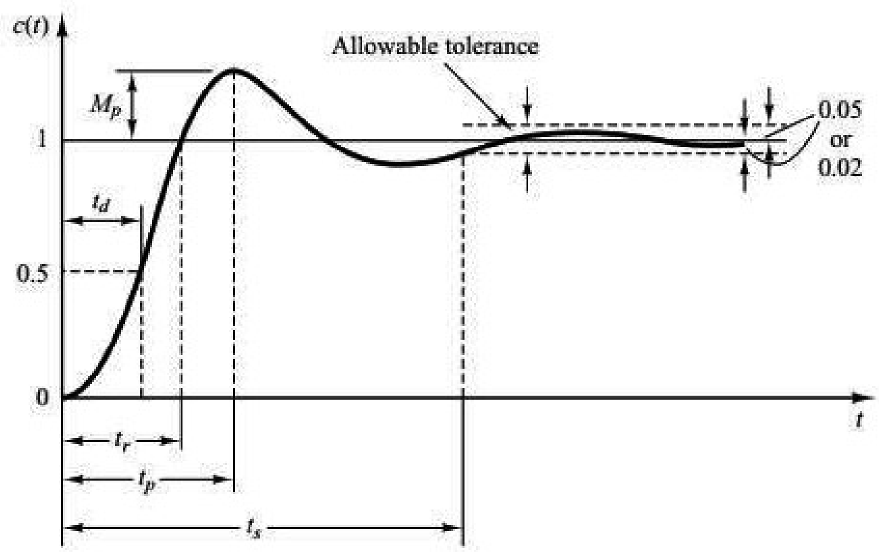


Fig.: Unit-step response curve showing td, tr, tp,Mp and ts.

For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

EXAMPLE:

Consider the system shown in Figure 5–6, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.

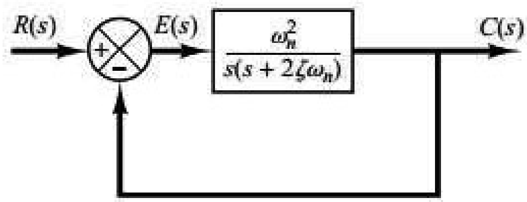


Figure 5–6
Second-order system.

Sol.

From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

$$E = \frac{1}{1+GD_{cl}}R = \mathcal{S}R, \quad \text{where} \quad \mathcal{S} = \frac{1}{1+GD_{cl}}. \qquad (4.27)$$

To consider polynomial inputs, we let $r(t) = t^k/k!1(t)$ for which the ransform is $R = \frac{1}{s^{k+1}}$. We take a mechanical system as the basis for a generic eference nomenclature, calling step inputs for which $k = 0$ "position" nputs, ramp inputs for which $k = 1$ are called "velocity" inputs, and if $: = 2$, the inputs are called "acceleration" inputs, regardless of the units of the actual signals. Application of the Final Value Theorem to the error ormula gives the result

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s), \qquad (4.28)$$

$$= \lim_{s\to 0} s\frac{1}{1+GD_{cl}}R(s), \qquad (4.29)$$

$$= \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s^{k+1}}. \qquad (4.30)$$

We consider first a system for which GD_{cl} has no pole at the origin, that is, no integrator, and a unit-step input for which $R(s) = 1/s$. Thus $r(t)$ is a polynomial of degree 0. In this case, Eq. (4.30) reduces to

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s}, \qquad (4.31)$$

$$\frac{e_{ss}}{r_{ss}} = \frac{e_{ss}}{1} = e_{ss} = \frac{1}{1+GD_{cl}(0)}, \qquad (4.32)$$

where $r_{ss} = \lim_{t\to\infty} r(t) = 1$. We define this system to be *Type 0* and we define the constant, $GD_{cl}(0) \triangleq K_p$, as the "*position error constant*." Notice that the above equation yields the relative error and if the input should be a polynomial of degree higher than 1, the resulting error would grow without bound. A polynomial of degree 0 is the highest degree a system of *Type 0* can track at all. If $GD_{cl}(s)$ has one pole at the origin, we could continue this line of argument and consider first-degree polynomial inputs but it is

4.2 Control of Steady-State Error to Polynomial Inputs: System Type

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Using Eq. (4.33), these results can be summarized by the foll equations:

$$K_p = \lim_{s\to 0} GD_{cl}(s), \quad n = 0,$$

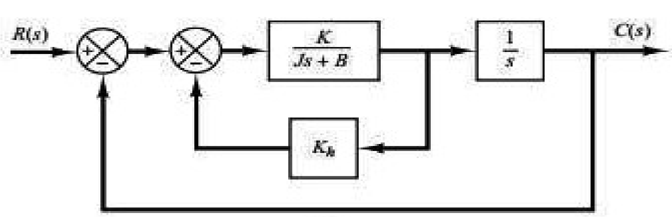
$$K_v = \lim_{s\to 0} sGD_{cl}(s), \quad n = 1,$$

$$K_a = \lim_{s\to 0} s^2 GD_{cl}(s), \quad n = 2.$$

5. Post- Test

1. For the system shown in Figure 5–13(a), determine the values of gain K and velocity-feedback constant Kh so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and Kh, obtain the rise time and settling time. Assume that J=1 kg-m2 and B=1 N-m/rad/sec. Determination of the values of K and Kh: The maximum overshoot Mp is given by Equation

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$



(a)

FIG. 5-13

Answers:

Answer Keys
Pre- Test
1. True 2. False, Input not output
Self-Test

1.

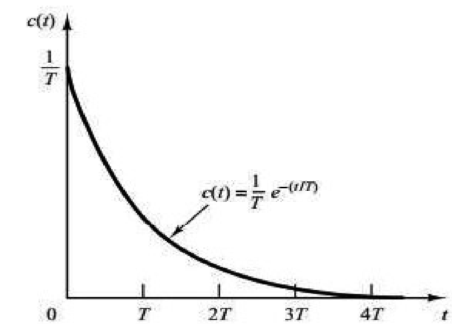
Unit-Impulse Response of First-Order Systems. For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1}$$

The inverse Laplace transform of Equation (5–7) gives

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0$$

The response curve given by Equation (5–8) is shown in Figure 5–4.



Post- Test

This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

The peak time t_p is specified as 1 sec; therefore, from Equation (5–20),

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then K_h is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Rise time t_r: From Equation (5–19), the rise time t_r is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$$

Thus, t_r is

$$t_r = 0.65 \text{ sec}$$

Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$