4.3 Higher Order Linear Differential Equation

The general form:

$$a_o\, y^n + a_1\, y^{n-1} + a_2\, y^{n-2} \dots\dots\dots\dots\dots\dots\dots\, a_{n-1}\acute{y} \ + a_n y = R(x)$$

تحل هذه المعادلة بنفس أسلوب حل المعادلة من الرتبة الثانية، فحلها تتكون من جزئيين (حل معادلة المتجانسة(y_c) و(حل معادلة الغير متجانسة (Y)).

لحل هذه المعادلة المتجانسة تستخدم أيضا المعادلة المميزة:

$$a_o\, m^n + a_1\, m^{n-1} + +a_2\, m^{n-2} \dots\dots\dots\dots\dots\dots\dots\, a_{n-1}m + a_n m = 0$$

وعند حل المعادلة المميزة نحصل على (n) من الجذور وبذلك نحصل على (n) من الحلول.

$$y_c = c_1\, y_1 + c_2\, y_2 + c_3\, y_3 + \ \dots\dots\dots\dots\dots\dots . + c_n\, y_n$$

وعند ظهور جذور متشابهة، يثبت احد هذه الجذور وتضرب الأخرى ب(x^n) للتخلص من التشابه، حيث n أصغر عدد صحيح موجب يزيل التكرار.

Example (1): Find the complete solution of the differential equation:

$$y''' + 5y'' + 9\,y' + 5y = 3\, e^{2x}$$

Solve:

$$y''' + 5y'' + 9\,y' + 5y = 0$$

$$m^3 + 5m^2 + 9\,m + 5 = 0$$

by try and error get to:

$let\ m = 1 \quad \rightarrow$

$1 + 5 + 9 + 5 = 20 \neq 0 \qquad \therefore not\ ok$

$let\ m = -1 \quad \rightarrow \quad -1 + 5 - 9 + 5 = 0 \qquad \therefore ok$

$$\begin{array}{r|l}
 & m^2 + 4\,m + 5 \\
\hline
\boxed{m+1} & \boxed{m^3 + 5m^2 + 9\,m + 5} \\
 & m^3 + m^2 \\
\hline
 & 4m^2 + 9\,m \\
 & 4m^2 + 4\,m \\
\hline
 & 5\,m + 5 \\
 & 5\,m + 5 \\
\hline
 & 0
\end{array}$$

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$$(m+1)(m^2+4\,m+5)=0$$

$$m+1=0 \quad \rightarrow \quad m_1=-1$$

$$m^2+4\,m+5=0 \quad \rightarrow$$

$$m_{2,3}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$m_{2,3}=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$$

$$y_c=c_1\,e^{mx}+e^{Px}(c_2\cos qx+c_3\sin qx)$$

$$y_c=c_1\,e^{-x}+e^{-2x}(c_2\cos x+c_3\sin x)$$

$$Let\;\; Y=A\,e^{2x}$$

$$Y'=2A\,e^{2x} \qquad Y''=4A\,e^{2x}$$

$$Y'''=8A\,e^{2x}$$

$$8A\,e^{2x}+20A\,e^{2x}+18A\,e^{2x}+5A\,e^{2x}=3\,e^{2x}$$

$$51A\,e^{2x}=3\,e^{2x} \quad \rightarrow \quad A=\frac{3}{51}$$

$$Y=\frac{3}{51}\,e^{2x}$$

∴ The general solution: $y=c_1\,e^{-x}+e^{-2x}(c_2\;\cos x+c_3\sin x)+\dfrac{3}{51}\,e^{2x}$

Example (2): Find the complete solution of the differential equation:

$$\boldsymbol{(D^4+8D^2+16)y=\;-\sin x}$$

Solve:

$$m^4+8m^2+16=0$$

$$(m^2+4)(m^2+4)=0 \quad \rightarrow$$

$$m^2+4=0 \quad \rightarrow \quad m^2=-4 \qquad m_{1,2}=\pm 2i$$

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$m^2 + 4 = 0 \quad \rightarrow \quad m^2 = -4$

$m_{3,4} = \pm 2i$

$y_{c1} = e^{Px}(c_1 \cos qx + c_2 \sin qx) = c_1 \cos 2x + c_2 \sin 2x$

$y_{c2} = e^{Px}(c_3 \cos qx + c_4 \sin qx) = c_3 x \cos 2x + c_4 x \sin 2x$

$y_c = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$

$Let \;\; Y = A \sin x + B \cos x$

$Y' = A \cos x - B \sin x \qquad Y'' = -A \sin x - B \cos x$

$Y''' = -A \cos x + B \sin x \qquad Y'''' = A \sin x + B \cos x$

$A \sin x + B \cos x + 8(-A \sin x - B \cos x) + 16(A \sin x + B \cos x) = -\sin x$

$9A \; \sin x + 9B \cos x = -\sin x \qquad \rightarrow \qquad A = -\frac{1}{9}, \quad B = 0$

$Y = -\frac{1}{9} \sin x$

$y = y_c + Y$

$\therefore$ The general solution: $y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x - \frac{1}{9} \sin x$

Example (3): Find the complete solution of the differential equation:

$\boldsymbol{(D^4 + 3D^3 + 3D^2 + D)y = \; 2x + 8}$

Solve:

$m^4 + 3m^3 + 3m^2 + m = 0$

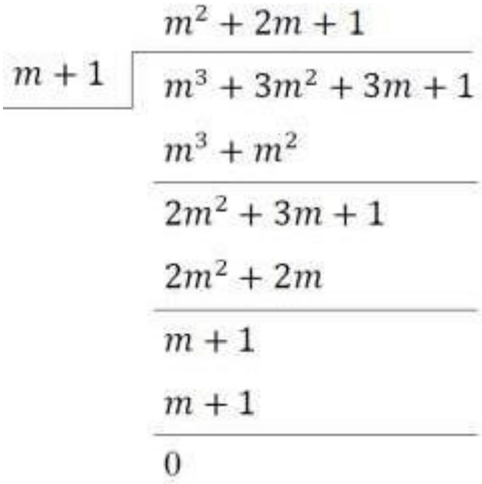
$m \; (m^3 + 3m^2 + 3m + 1) = 0$

$m_1 = 0 \qquad\qquad m^3 + 3m^2 + 3m + 1 = 0$

by try and error get to:

$let\ m = -1 \quad \rightarrow \quad -1 + 3 - 3 + 1 = 0 \qquad \therefore ok$

$(m + 1)(m^2 + 2m + 1) = 0$



$$\begin{array}{r|l} & m^2 + 2m + 1 \\ \hline m + 1 & m^3 + 3m^2 + 3m + 1 \\ & m^3 + m^2 \\ \hline & 2m^2 + 3m + 1 \\ & 2m^2 + 2m \\ \hline & m + 1 \\ & m + 1 \\ \hline & 0 \end{array}$$

$(m + 1)(m + 1)(m + 1) = 0$

$m_{2,3,4} = -1$

$y_c = c_1 e^0 + c_2\, e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x}$

$Let\ \ Y = Ax^2 + Bx$

$\acute{Y} = 2Ax + B$

$\acute{Y} = 2A, \quad \acute{Y} = 0, \quad \acute{Y} = 0$

$6A + 2Ax + B = 2x + 8 \qquad \rightarrow$

$2A = 2 \qquad \rightarrow \qquad A = 1$

$6A + B = 8 \qquad \rightarrow \qquad B = 2$

$Y = x^2 + 2x \qquad y = y_c + Y$

$\therefore$ The complete solution: $y = c_1 e^0 + c_2\, e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x} + x^2 + 2x$

Example (4): Find the general solution of the differential equation:

$\boldsymbol{(D^4 - 16)y = 0}$

Solve:

$m^4 - 16 = 0$

$(m^2 - 4)(m^2 + 4) = 0 \quad \rightarrow$

$m^2 - 4 = 0 \quad \rightarrow \quad m^2 = 4 \quad \rightarrow \quad m_{1,2} = \pm 2 \quad m_1 \neq m_2$

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$m^2 + 4 = 0 \quad \rightarrow \quad m^2 = -4 \quad \rightarrow \quad m_{3,4} = \pm 2i \quad m_3, m_4$ Imaginary

$$y_c = c_1 e^{m_1 x} + c_2\, e^{m_2 x} + e^{Px}(c_3 \cos qx + c_4 \sin qx)$$

$$y_c = c_1 e^{2x} + c_2\, e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

Example (5): Find the general solution of the differential equation:

$$\boldsymbol{(D^8 - 2D^4 + 1)y = 0}$$

Solve:

$$m^8 - 2m^4 + 1 = 0$$

$$(m^4 - 1)(m^4 - 1) = 0 \quad \rightarrow$$

$$(m^2 - 1)(m^2 + 1)(m^2 - 1)(m^2 + 1) = 0$$

$$(m - 1)(m + 1)(m^2 + 1)(m - 1)(m + 1)(m^2 + 1) = 0$$

$m - 1 = 0 \quad \rightarrow \quad m_1 = 1 \qquad m + 1 = 0 \rightarrow \quad m_2 = -1$

$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1 \quad \rightarrow \quad m_{3,4} = \pm i$

$m - 1 = 0 \quad \rightarrow \quad m_5 = 1$

$m + 1 = 0 \quad \rightarrow \quad m_6 = -1$

$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1 \quad \rightarrow \quad m_{7,8} = \pm i$

$m_1 = m_5 = 1$

$m_2 = m_6 = -1$

$m_{3,4} = m_{7,8} = \pm i$

$$y_c = c_1 e^{x} + c_2\, e^{-x} + c_3 \cos x + c_4 \sin x + c_5 x e^{x} + c_6\, x e^{-x} + c_7 x \cos x + c_8 x \sin x$$

H.W: Find the complete solution of the differential equation:

1) $\acute{y} - 2\acute{y} - 3\,\acute{y} + 10y = 40\ \cos x\,, \quad let\ m = -2$

Ans: $y = c_1 e^{-2x} + e^{2x}(c_2 \cos 2x + c_3 \sin 2x) + 3\ \cos x - \sin x$

2) $(D^4 + 8D^2 - 9)y = \ 9x^2$

Ans: $y = c_1 e^x + c_2 e^{-x} + c_3 \cos 3x + c_4 \sin 3x + x^2 + 16/9$

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