Application of Ordinary Differential equations:

Ex.1/ A metal bar at Temp. $100^\circ F$ is placed in a room at temp. $0^\circ F$. If the temp. of the bar is proportional to the difference between the temp. of the body and the surrounding media. After 20 min the temp. of the bar is $50^\circ F$. Find the temp at any time?

Sol:-

$$\frac{dT}{dt} \propto (T_{body} - T_{media}) \rightarrow \frac{dT}{dt} = K(T_b - \cancel{T_m}^{0})$$

$$\frac{dT}{dt} = kT_b \rightarrow \int \frac{dT}{T_b} = \int k\,dt$$

$$\ln T = Kt + C$$

$$T(t) = Ce^{kt} \quad \text{--- } 1$$

B.C.1 at time $= 0$ $\rightarrow T_b = 100^\circ F$

$$100 = Ce^{0} \rightarrow C = 100$$

$$\therefore T(t) = 100e^{kt}$$

B.C.2 at time $= 20$ min $\rightarrow T_b = 50^\circ F$

$$50 = 100e^{20K}$$

$$0.5 = e^{20K}$$

$$\therefore K = -0.0346$$

$$\Rightarrow T(t) = 100e^{-0.0346t} \rightarrow \text{The eq. of temp. at any time}$$

EX.2/ A population of a small town grows proportion to it's current population. The initial population is 5000 and grows 4% per year.

a) Find an equation to model the population.

b) Determine the population after 3 years.

c) Determine how long it will take the population to double.

Sol/

a) $\frac{dP}{dt} = 0.04P$, $P(0) = 5000$

$dP = 0.04P\,dt \Rightarrow \int \frac{1}{P}\,dP = \int 0.04\,dt$

$\ln P = 0.04t + C \Rightarrow P(t) = e^{0.04t} \cdot e^{c}$

$P(t) = P_0 e^{0.04t}$

B.C at $t = 0$, $P_0 = 5000$

$\Rightarrow P(t) = 5000\,e^{0.04t}$

b) $t = 3$ years

$P(3) = 5000\,e^{0.04(3)}$

$P(3) = 5637$ capita

c) $t = ??$, $P = 10000$

$$P(t) = 5000\, e^{0.04t}$$

$$\frac{10000}{5000} = \frac{5000}{5000} e^{0.04t}$$

$$2 = e^{0.04t}$$

$$\ln 2 = 0.04t$$

$$0.69314 = 0.04t$$

$$\therefore t = 17.3 \text{ years}$$

H.W3/ Repeat example 1 with $T_{media} = 20\,F$.

EX-3 / water at 190 °F is left in a room of 70 °F.
At time = 0 ; the water is cooling at 15°F per minute.
a) Find the Function that models the cooling of water.
b) How long will it take for the temp. to reach 143 °F?

sol:-

a) Newton's cooling law :- $\frac{dT}{dt} = k\,(T_w - T_r)$

$\frac{dT}{dt} = -15$; at $t=0$ $T_w = 190$ and $T_r = 70$

$-15 = k(190 - 70) \Rightarrow k = -0.125$

$\therefore \frac{dT}{dt} = -0.125\,(T_w - T_c)$

$dT = -0.125\,(T_w - 70)\,dt$

$\int \frac{1}{T-70}\,dT = \int -0.125\,dt$

$\ln(T-70) = -0.125t + c$

$T-70 = e^{-0.125+c} \longrightarrow T-70 = e^{-0.125t} \cdot e^{c}$ $\boxed{e^{c} = a}$

$\therefore T(t) = a\,e^{-0.125t} + 70$

<u>B.C.1</u> At $t=0$, $T = 190°F$

$190 = a\,e^{-0.125(0)} + 70$

$\therefore a = 120 \Longrightarrow \boxed{T(t) = 120\,e^{-0.125t} + 70}$

b) $T = 143$; $t = ??$

$$143 = 120\, e^{-0.125t} + 70$$

$$143 - 70 = 120\, e^{-0.125t}$$

$$\frac{73}{120} = e^{-0.125t}$$

$$0.608333 = e^{-0.125t}$$

$$-0.49703 = -0.125\, t$$

$$\therefore t = 3.976 \text{ min}$$

EX.4/ A copper ball is heated at a temp. 100°C. Then at time t=0 it is placed in a water which is maintained at a temp. of 30°C. At the end of 3min the temp of the ball is reduced to 70°C. Find the time at which the temp of the ball is reduced to 31°C.

Sol/

$$\frac{dT}{dt} = K(T-c) \Longrightarrow \frac{dT}{dt} = K(T-30)$$

$$\frac{dT}{dt} - kT = -30k \Longrightarrow \text{linear eq.} \rightarrow P(x) = -k \; ; \; Q(x) = -30k$$

$$I.f = e^{\int -k\,dt} \rightarrow I.f = e^{-kt}$$

$$T(t) = \frac{1}{e^{-kt}} \int e^{-kt} * -30k\, dt + \frac{c}{e^{-kt}}$$

$$= \frac{1}{\cancel{e^{-kt}}} (30\cancel{e^{-kt}}) + \frac{c}{e^{-kt}}$$

$$T(t) = 30 + Ce^{kt} \qquad (\text{sub. B.C1 at } t=0, T=100)$$

$$100 = 30 + Ce^{k(0)} \Longrightarrow C = 70$$

$$\therefore T = 30 + 70e^{kt}$$

B.C.2 At time = 3min, T = 70°C

$$70 = 30 + 70e^{k(3)}$$

$$k = -0.1865$$

$$\therefore T = 30 + 70\, e^{-0.1865 t}$$

Determine t at $T = 31^{\circ}C$

$$31 = 30 + 70\, e^{-0.1865t}$$

$$31 - 30 = 70\, e^{-0.1865t}$$

$$\frac{1}{70} = e^{-0.1865t}$$

$$0.01428 = e^{-0.1865t}$$

$$-4.2488 = -0.1865t$$

$$\therefore t = 22.78$$