$x < 0$ is missing. The reason the graphs differ is that many calculators and computer software programs calculate $x^{1/3}$ as $e^{(1/3)\ln x}$. Since the logarithmic function is not defined for negative values of x, the computing device can produce only the right branch, where $x > 0$. (Logarithmic and exponential functions are introduced in the next two sections.)

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals $x^{1/3}$ except at $x = 0$ (where f is undefined, although $0^{1/3} = 0$). The graph of f is shown in Figure 1.55b. ■

Exercises 1.4

Choosing a Viewing Window

T In Exercises 1–4, use a graphing calculator or computer to determine which of the given viewing windows displays the most appropriate graph of the specified function.

1. $f(x) = x^4 - 7x^2 + 6x$

a. $[-1, 1]$ by $[-1, 1]$ **b.** $[-2, 2]$ by $[-5, 5]$
c. $[-10, 10]$ by $[-10, 10]$ **d.** $[-5, 5]$ by $[-25, 15]$

2. $f(x) = x^3 - 4x^2 - 4x + 16$

a. $[-1, 1]$ by $[-5, 5]$ **b.** $[-3, 3]$ by $[-10, 10]$
c. $[-5, 5]$ by $[-10, 20]$ **d.** $[-20, 20]$ by $[-100, 100]$

3. $f(x) = 5 + 12x - x^3$

a. $[-1, 1]$ by $[-1, 1]$ **b.** $[-5, 5]$ by $[-10, 10]$
c. $[-4, 4]$ by $[-20, 20]$ **d.** $[-4, 5]$ by $[-15, 25]$

4. $f(x) = \sqrt{5 + 4x - x^2}$

a. $[-2, 2]$ by $[-2, 2]$ **b.** $[-2, 6]$ by $[-1, 4]$
c. $[-3, 7]$ by $[0, 10]$ **d.** $[-10, 10]$ by $[-10, 10]$

Finding a Viewing Window

T In Exercises 5–30, find an appropriate viewing window for the given function and use it to display its graph.

5. $f(x) = x^4 - 4x^3 + 15$ **6.** $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$

7. $f(x) = x^5 - 5x^4 + 10$ **8.** $f(x) = 4x^3 - x^4$

9. $f(x) = x\sqrt{9 - x^2}$ **10.** $f(x) = x^2(6 - x^3)$

11. $y = 2x - 3x^{2/3}$ **12.** $y = x^{1/3}(x^2 - 8)$

13. $y = 5x^{2/5} - 2x$ **14.** $y = x^{2/3}(5 - x)$

15. $y = |x^2 - 1|$ **16.** $y = |x^2 - x|$

17. $y = \frac{x + 3}{x + 2}$ **18.** $y = 1 - \frac{1}{x + 3}$

19. $f(x) = \frac{x^2 + 2}{x^2 + 1}$ **20.** $f(x) = \frac{x^2 - 1}{x^2 + 1}$

21. $f(x) = \frac{x - 1}{x^2 - x - 6}$ **22.** $f(x) = \frac{8}{x^2 - 9}$

23. $f(x) = \frac{6x^2 - 15x + 6}{4x^2 - 10x}$ **24.** $f(x) = \frac{x^2 - 3}{x - 2}$

25. $y = \sin 250x$ **26.** $y = 3\cos 60x$

27. $y = \cos\left(\frac{x}{50}\right)$ **28.** $y = \frac{1}{10}\sin\left(\frac{x}{10}\right)$

29. $y = x + \frac{1}{10}\sin 30x$ **30.** $y = x^2 + \frac{1}{50}\cos 100x$

31. Graph the lower half of the circle defined by the equation $x^2 + 2x = 4 + 4y - y^2$.

32. Graph the upper branch of the hyperbola $y^2 - 16x^2 = 1$.

33. Graph four periods of the function $f(x) = -\tan 2x$.

34. Graph two periods of the function $f(x) = 3\cot\frac{x}{2} + 1$.

35. Graph the function $f(x) = \sin 2x + \cos 3x$.

36. Graph the function $f(x) = \sin^3 x$.

Graphing in Dot Mode

T Another way to avoid incorrect connections when using a graphing device is through the use of a "dot mode," which plots only the points. If your graphing utility allows that mode, use it to plot the functions in Exercises 37–40.

37. $y = \frac{1}{x - 3}$ **38.** $y = \sin\frac{1}{x}$

39. $y = x\lfloor x \rfloor$ **40.** $y = \frac{x^3 - 1}{x^2 - 1}$

1.5 Exponential Functions

Exponential functions are among the most important in mathematics and occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of a disease, consumption of natural resources, the earth's atmospheric pressure, temperature change of a heated object placed in a cooler environment, and the dating of fossils. In this section we introduce these functions informally, using an intuitive approach. We give a rigorous development of them in Chapter 7, based on important calculus ideas and results.

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to the consideration of the function $f(x) = 2^x$. We call this an *exponential* function because the variable x appears in the exponent of 2^x. Functions such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function

Don't confuse 2^x with the power x^2, where the variable x is the base, not the exponent.

$$f(x) = a^x$$

is the **exponential function with base *a***.

EXAMPLE 1 In 2000, \$100 is invested in a savings account, where it grows by accruing interest that is compounded annually (once a year) at an interest rate of 5.5%. Assuming no additional funds are deposited to the account and no money is withdrawn, give a formula for a function describing the amount A in the account after x years have elapsed.

Solution If $P = 100$, at the end of the first year the amount in the account is the original amount plus the interest accrued, or

$$P + \left(\frac{5.5}{100}\right)P = (1 + 0.055)P = (1.055)P.$$

At the end of the second year the account earns interest again and grows to

$$(1 + 0.055)\cdot(1.055P) = (1.055)^2P = 100\cdot(1.055)^2. \qquad P = 100$$

Continuing this process, after x years the value of the account is

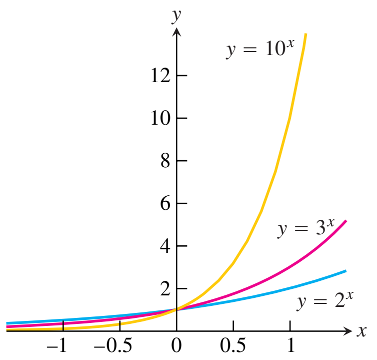
$$A = 100\cdot(1.055)^x.$$

This is a multiple of the exponential function with base 1.055. Table 1.4 shows the amounts accrued over the first four years. Notice that the amount in the account each year is always 1.055 times its value in the previous year.

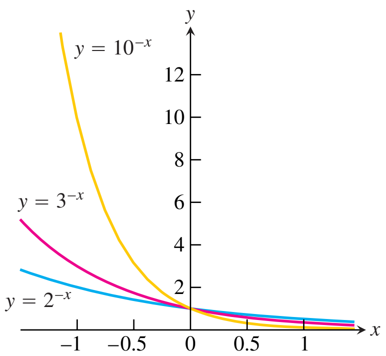
TABLE 1.4 Savings account growth

Year	Amount (dollars)	Increase (dollars)
2000	100	
2001	$100(1.055) = 105.50$	5.50
2002	$100(1.055)^2 = 111.30$	5.80
2003	$100(1.055)^3 = 117.42$	6.12
2004	$100(1.055)^4 = 123.88$	6.46

In general, the amount after x years is given by $P(1 + r)^x$, where r is the interest rate (expressed as a decimal). ■



(a) $y = 2^x$, $y = 3^x$, $y = 10^x$



(b) $y = 2^{-x}$, $y = 3^{-x}$, $y = 10^{-x}$

FIGURE 1.56 Graphs of exponential functions.

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically as follows. If $x = n$ is a positive integer, the number a^n is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$ for some positive integer n, then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n.$$

If $x = 1/n$ for some positive integer n, then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a. If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

If x is *irrational*, the meaning of a^x is not so clear, but its value can be defined by considering values for rational numbers that get closer and closer to x. This informal approach is based on the graph of the exponential function. In Chapter 7 we define the meaning in a rigorous way.

We displayed the graphs of several exponential functions in Section 1.1, and show them again here in Figure 1.56. These graphs describe the values of the exponential functions for all real inputs x. The value at an irrational number x is chosen so that the graph of a^x has no "holes" or "jumps." Of course, these words are not mathematical terms, but they do convey the informal idea. We mean that the value of a^x, when x is irrational, is chosen so that the function $f(x) = a^x$ is *continuous*, a notion that will be carefully explored in the next chapter. This choice ensures the graph retains its increasing behavior when $a > 1$, or decreasing behavior when $0 < a < 1$ (see Figure 1.56).

Arithmetically, the graphical idea can be described in the following way, using the exponential $f(x) = 2^x$ as an illustration. Any particular irrational number, say $x = \sqrt{3}$, has a decimal expansion

$$\sqrt{3} = 1.732050808 \ldots.$$

We then consider the list of numbers, given as follows in the order of taking more and more digits in the decimal expansion,

$$2^1, 2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \ldots. \tag{1}$$

We know the meaning of each number in list (1) because the successive decimal approximations to $\sqrt{3}$ given by 1, 1.7, 1.73, 1.732, and so on, are all *rational* numbers. As these decimal approximations get closer and closer to $\sqrt{3}$, it seems reasonable that the list of numbers in (1) gets closer and closer to some fixed number, which we specify to be $2^{\sqrt{3}}$.

Table 1.5 illustrates how taking better approximations to $\sqrt{3}$ gives better approximations to the number $2^{\sqrt{3}} \approx 3.321997086$. It is the *completeness property* of the real numbers (discussed briefly in Appendix 6) which guarantees that this procedure gives a single number we define to be $2^{\sqrt{3}}$ (although it is beyond the scope of this text to give a proof). In a similar way, we can identify the number 2^x (or a^x, $a > 0$) for any irrational x. By identifying the number a^x for both rational and irrational x, we eliminate any "holes" or "gaps" in the graph of a^x. In practice you can use a calculator to find the number a^x for irrational x, taking successive decimal approximations to x and creating a table similar to Table 1.5.

Exponential functions obey the familiar rules of exponents listed on the next page. It is easy to check these rules using algebra when the exponents are integers or rational numbers. We prove them for all real exponents in Chapters 4 and 7.

TABLE 1.5 Values of $2^{\sqrt{3}}$ for rational r closer and closer to $\sqrt{3}$

r	2^r
1.0	2.000000000
1.7	3.249009585
1.73	3.317278183
1.732	3.321880096
1.7320	3.321880096
1.73205	3.321995226
1.732050	3.321995226
1.7320508	3.321997068
1.73205080	3.321997068
1.732050808	3.321997086

Rules for Exponents

If $a > 0$ and $b > 0$, the following rules hold true for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

EXAMPLE 2 We illustrate using the rules for exponents.

1. $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$
2. $\dfrac{\left(\sqrt{10}\right)^3}{\sqrt{10}} = \left(\sqrt{10}\right)^{3-1} = \left(\sqrt{10}\right)^2 = 10$
3. $\left(5^{\sqrt{2}}\right)^{\sqrt{2}} = 5^{\sqrt{2}\cdot\sqrt{2}} = 5^2 = 25$
4. $7^{\pi} \cdot 8^{\pi} = (56)^{\pi}$
5. $\left(\dfrac{4}{9}\right)^{1/2} = \dfrac{4^{1/2}}{9^{1/2}} = \dfrac{2}{3}$ ■

The Natural Exponential Function e^x

The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e. The number e is irrational, and its value is 2.718281828 to nine decimal places. It might seem strange that we would use this number for a base rather than a simple number like 2 or 10. The advantage in using e as a base is that it simplifies many of the calculations in calculus.

If you look at Figure 1.56a you can see that the graphs of the exponential functions $y = a^x$ get steeper as the base a gets larger. This idea of steepness is conveyed by the slope of the tangent line to the graph at a point. Tangent lines to graphs of functions are defined precisely in the next chapter, but intuitively the tangent line to the graph at a point is a line that just touches the graph at the point, like a tangent to a circle. Figure 1.57 shows the slope of the graph of $y = a^x$ as it crosses the y-axis for several values of a. Notice that the slope is exactly equal to 1 when a equals the number e. The slope is smaller than 1 if $a < e$, and larger than 1 if $a > e$. This is the property that makes the number e so useful in calculus: **The graph of $y = e^x$ has slope 1 when it crosses the y-axis**.

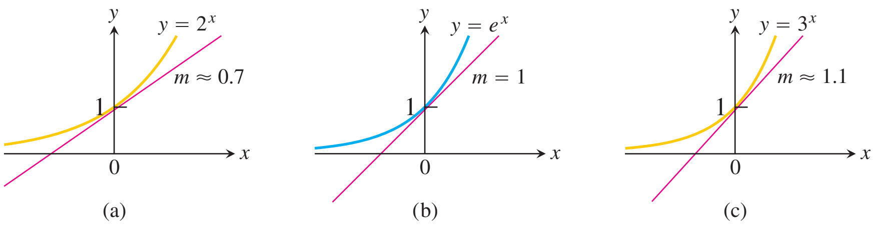


FIGURE 1.57 Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y-axis. The slope is smaller for a base less than e, such as 2^x, and larger for a base greater than e, such as 3^x.

In Chapter 3 we use that slope property to prove e is the number the quantity $(1 + 1/x)^x$ approaches as x becomes large without bound. That result provides one way to compute the value of e, at least approximately. The graph and table in Figure 1.58 show the behavior of this expression and how it gets closer and closer to the line $y = e \approx 2.718281828$ as x gets larger and larger. (This *limit* idea is made precise in the next chapter.) A more complete discussion of e is given in Chapter 7.

x	$(1 + 1/x)^x$
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.7180
6000	2.7181
7000	2.7181

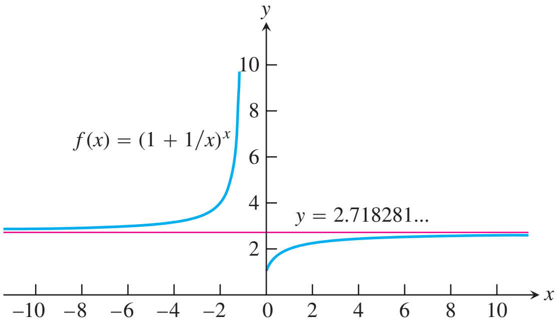


FIGURE 1.58 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as x gets larger and larger, $f(x)$ gets closer and closer to $e \approx 2.7182818\ldots$.

Exponential Growth and Decay

The exponential functions $y = e^{kx}$, where k is a nonzero constant, are frequently used for modeling exponential growth or decay. The function $y = y_0 e^{kx}$ is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$. Here y_0 represents a constant. An example of exponential growth occurs when computing interest **compounded continuously** modeled by $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in units consistent with r. An example of exponential decay is the model $y = A \cdot e^{-1.2 \times 10^{-4} t}$, which represents how the radioactive element carbon-14 decays over time. Here A is the original amount of carbon-14 and t is the time in years. Carbon-14 decay is used to date the remains of dead organisms such as shells, seeds, and wooden artifacts. Figure 1.59 shows graphs of exponential growth and exponential decay.

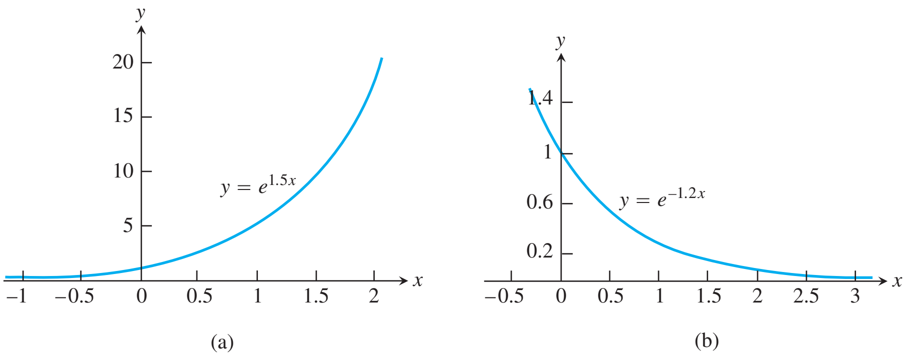


FIGURE 1.59 Graphs of (a) exponential growth, $k = 1.5 > 0$, and (b) exponential decay, $k = -1.2 < 0$.

EXAMPLE 3 Investment companies often use the model $y = Pe^{rt}$ in calculating the growth of an investment. Use this model to track the growth of \$100 invested in 2000 at an annual interest rate of 5.5%.

Solution Let $t = 0$ represent 2000, $t = 1$ represent 2001, and so on. Then the exponential growth model is $y(t) = Pe^{rt}$, where $P = 100$ (the initial investment), $r = 0.055$ (the annual interest rate expressed as a decimal), and t is time in years. To predict the amount in the account in 2004, after four years have elapsed, we take $t = 4$ and calculate

$$\begin{aligned} y(4) &= 100e^{0.055(4)} \\ &= 100e^{0.22} \\ &= 124.61. \qquad \text{Nearest cent using calculator} \end{aligned}$$

This compares with \$123.88 in the account when the interest is compounded annually from Example 1. ■

EXAMPLE 4 Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder of the atom re-forming to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium eventually decays into lead. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-rt}, \qquad r > 0.$$

The number r is called the **decay rate** of the radioactive substance. (We will see how this formula is obtained in Section 7.2.) For carbon-14, the decay rate has been determined experimentally to be about $r = 1.2 \times 10^{-4}$ when t is measured in years. Predict the percent of carbon-14 present after 866 years have elapsed.

Solution If we start with an amount y_0 of carbon-14 nuclei, after 866 years we are left with the amount

$$\begin{aligned} y(866) &= y_0 e^{(-1.2\times10^{-4})(866)} \\ &\approx (0.901)y_0. \qquad \text{Calculator evaluation} \end{aligned}$$

That is, after 866 years, we are left with about 90% of the original amount of carbon-14, so about 10% of the original nuclei have decayed. In Example 7 in the next section, you will see how to find the number of years required for half of the radioactive nuclei present in a sample to decay (called the *half-life* of the substance). ■

You may wonder why we use the family of functions $y = e^{kx}$ for different values of the constant k instead of the general exponential functions $y = a^x$. In the next section, we show that the exponential function a^x is equal to e^{kx} for an appropriate value of k. So the formula $y = e^{kx}$ covers the entire range of possibilities, and we will see that it is easier to use.

Exercises 1.5

Sketching Exponential Curves

In Exercises 1–6, sketch the given curves together in the appropriate coordinate plane and label each curve with its equation.

1. $y = 2^x, y = 4^x, y = 3^{-x}, y = (1/5)^x$
2. $y = 3^x, y = 8^x, y = 2^{-x}, y = (1/4)^x$
3. $y = 2^{-t}$ and $y = -2^t$
4. $y = 3^{-t}$ and $y = -3^t$
5. $y = e^x$ and $y = 1/e^x$
6. $y = -e^x$ and $y = -e^{-x}$

In each of Exercises 7–10, sketch the shifted exponential curves.

7. $y = 2^x - 1$ and $y = 2^{-x} - 1$
8. $y = 3^x + 2$ and $y = 3^{-x} + 2$
9. $y = 1 - e^x$ and $y = 1 - e^{-x}$
10. $y = -1 - e^x$ and $y = -1 - e^{-x}$

Applying the Laws of Exponents

Use the laws of exponents to simplify the expressions in Exercises 11–20.

11. $16^2 \cdot 16^{-1.75}$
12. $9^{1/3} \cdot 9^{1/6}$
13. $\dfrac{4^{4.2}}{4^{3.7}}$
14. $\dfrac{3^{5/3}}{3^{2/3}}$
15. $\left(25^{1/8}\right)^4$
16. $\left(13^{\sqrt{2}}\right)^{\sqrt{2}/2}$
17. $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$
18. $\left(\sqrt{3}\right)^{1/2} \cdot \left(\sqrt{12}\right)^{1/2}$
19. $\left(\dfrac{2}{\sqrt{2}}\right)^4$
20. $\left(\dfrac{\sqrt{6}}{3}\right)^2$

Composites Involving Exponential Functions

Find the domain and range for each of the functions in Exercises 21–24.

21. $f(x) = \dfrac{1}{2 + e^x}$

22. $g(t) = \cos(e^{-t})$

23. $g(t) = \sqrt{1 + 3^{-t}}$

24. $f(x) = \dfrac{3}{1 - e^{2x}}$

Applications

T In Exercises 25–28, use graphs to find approximate solutions.

25. $2^x = 5$

26. $e^x = 4$

27. $3^x - 0.5 = 0$

28. $3 - 2^{-x} = 0$

T In Exercises 29–36, use an exponential model and a graphing calculator to estimate the answer in each problem.

29. Population growth The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

30. Population growth The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

a. Estimate the population in 1915 and 1940.

b. Approximately when did the population reach 50,000?

31. Radioactive decay The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

a. Express the amount of phosphorus-32 remaining as a function of time t.

b. When will there be 1 gram remaining?

32. If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

33. Doubling your money Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

34. Tripling your money Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

35. Cholera bacteria Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hr?

36. Eliminating a disease Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

a. to reduce the number of cases to 1000?

b. to eliminate the disease; that is, to reduce the number of cases to less than 1?

1.6 Inverse Functions and Logarithms

A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.