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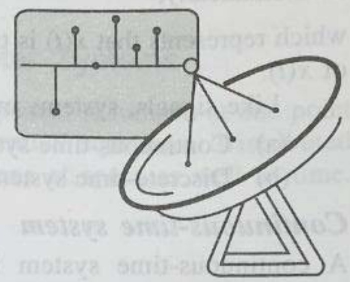

Systems

2.1 INTRODUCTION

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends upon the transfer function of the system. It is a cause-and-effect relation between two or more signals. There are various types of systems: electrical systems, mechanical systems, biological systems, opto-electronic systems, electromechanical systems and so on. The actual physical structure of the system determines the exact relation between the input $x(t)$ and the output $y(t)$ and specifies the output for every input. Physical devices such as motor, amplifier, filter, boiler and turbine are examples of systems. In this chapter, we discuss about classification of systems and determination of the type of system from the describing equation. Systems may be single input and single output systems or multi input and multi output systems. In this book, we consider only single input and single output systems.

2.2 CLASSIFICATION OF SYSTEMS

A system is represented by a block diagram as shown in Figure 2.1. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the input is denoted by $x(t)$ and the output is denoted by $y(t)$.

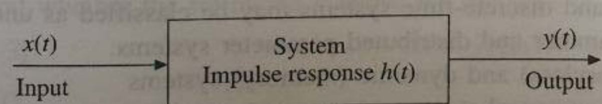

Figure 2.1 A system.

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The relation between the input $x(t)$ and the output $y(t)$ of a system has the form

$$y(t) = \text{Operation on } x(t)$$

Mathematically,

$$y(t) = T[x(t)]$$

which represents that $x(t)$ is transformed to $y(t)$. In other words $y(t)$ is the transformed version of $x(t)$.

Like signals, systems may also be broadly classified as under

(a) Continuous-time systems
(b) Discrete-time systems

Continuous-time system

A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals.

If the input and output of a continuous-time system are $x(t)$ and $y(t)$ then we can say that $x(t)$ is transformed to $y(t)$. That is,

$$y(t) = T[x(t)]$$

Amplifiers, filters, integrators, and differentiators are the examples of continuous-time systems. The block diagram of a continuous-time system is shown in Figure 2.2.

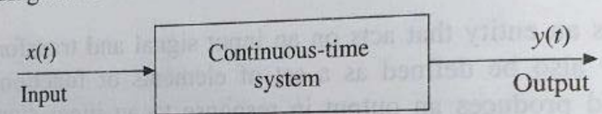

Figure 2.2 Block diagram of continuous-time system.

Discrete-time System

A discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

If the input and output of a discrete-time system are $x(n)$ and $y(n)$, then we can say that $x(n)$ is transformed to $y(n)$. That is,

$$y(n) = T[x(n)]$$

Microprocessors, semiconductor memories, shift registers, etc. are the examples of discrete-time systems.

The block diagram of a discrete-time system is shown in Figure 2.3.

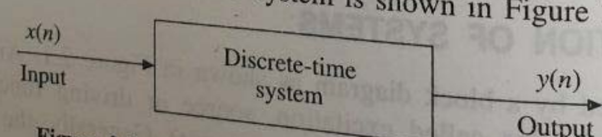

Figure 2.3 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems may be classified as under

1. Lumped parameter and distributed parameter systems
2. Static (memoryless) and dynamic (memory) systems
3. Causal and non-causal systems
4. Linear and non-linear systems
5. Time-invariant and time varying systems

6. Stable and unstable systems.
7. Invertible and non-invertible systems
8. FIR and IIR systems

2.2.1 Lumped Parameter and Distributed Parameter Systems

Lumped parameter systems are the systems in which each component is lumped at one point in space. These systems are described by ordinary differential equations. Distributed parameter systems are the systems in which signals are functions of space as well as time. These systems are described by partial differential equations.

2.2.2 Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e. for a static or memoryless system, the output at any instant t (or n) depends only on the input applied at that instant t (or n) but not on the past or future values of input.

For example, the systems defined below are static or memoryless systems.

$$y(t) = x(t)$$
$$y(t) = x^2(t)$$
$$y(n) = x(n)$$
$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs.

For example, the systems defined below are dynamic or memory systems.

$$y(t) = x(t-1)$$
$$y(t) = x(t) + x(t+2)$$
$$y(t) = \frac{d^2x(t)}{dt^2} + x(t)$$
$$y(n) = x(2n)$$
$$y(n) = x(n) + x(n-2)$$

Any continuous-time system described by a differential equation or any discrete-time system described by a difference equation is also a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A summer or accumulator is an example of a discrete-time system with memory. A delay is also a discrete-time system with memory.

EXAMPLE 2.1 Find whether the following systems are dynamic or not:

(a) $y(t) = x(t-3)$

(b) $y(t) = x(2t)$

(c) $y(t) = \frac{d^2x(t)}{dt^2} + 2x(t)$

(d) $y(n) = x(n+2)$

(e) $y(n) = x^2(n)$

(f) $y(n) = x(n-2) + x(n)$

Solution:

(a) Given $$y(t) = x(t-3)$$
The output depends on past value of input. Therefore, the system is dynamic.

(b) Given $$y(t) = x(2t)$$
The output depends on future value of input. Therefore, the system is dynamic.

(c) Given $$y(t) = \frac{d^2x(t)}{dt^2} + 2x(t)$$
The system is described by a differential equation. Therefore, the system is dynamic.

(d) Given $$y(n) = x(n+2)$$
The output depends on the future value of input. Therefore, the system is dynamic.

(e) Given $$y(n) = x^2(n)$$
The output depends on the present value of input alone. Therefore, the system is static.

(f) Given $$y(n) = x(n-2) + x(n)$$
The system is described by a difference equation. Therefore, the system is dynamic.

2.2.3 Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any time t depends only on the present and past values of the input but not on future inputs. In other words, we can say that a system is causal if the response or output does not begin before the input function is applied. That is, a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for t (or n) < 0, since $\delta(t)$ [or $\delta(n)$] exists only at t (or n) $= 0$.

i.e. $h(t) = 0$ for $t < 0$ and $h(n) = 0$ for $n < 0$

Examples for the causal systems are:

$$y(t) = x(t-2) + 2x(t)$$
$$y(t) = tx(t)$$
$$y(n) = nx(n)$$
$$y(n) = x(n-2) + x(n-1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any time t depends on future inputs. They do not exist in real time. They are not physically realizable. They are anticipatory systems. They produce an output even before the input is given.

Examples for the non-causal systems are:

$$y(t) = x(t+2) + x(t)$$
$$y(t) = x^2(t) + tx(t+1)$$
$$y(n) = x(n) + x(2n)$$
$$y(n) = x^2(n) + 2x(n+2)$$

A resistor is an example of a continuous-time causal system. Image processing systems are examples of non causal systems.

EXAMPLE 2.2 Check whether the following systems are causal or not:

(a) $y(t) = x^2(t) + x(t-4)$ (b) $y(t) = x(2-t) + x(t-4)$

(c) $y(t) = \int_{-\infty}^{3t} x(\tau)\,d\tau$ (d) $y(t) = x\left(\frac{t}{2}\right)$

(e) $y(t) = x[\sin 2t]$ (f) $y(n) = x(n) + x(n-2)$

(g) $y(n) = x(2n)$ (h) $y(n) = \sin[x(n)]$

(i) $y(n) = x(-n)$

Solution:

(a) Given $y(t) = x^2(t) + x(t-4)$

For $t = -2$ $y(-2) = x^2(-2) + x(-6)$

For $t = 0$ $y(0) = x^2(0) + x(-4)$

For $t = 2$ $y(2) = x^2(2) + x(-2)$

For all values of t, the output depends only on the present and past values of input. Therefore, the system is causal.

(b) Given $y(t) = x(2-t) + x(t-4)$

For $t = -1$ $y(-1) = x(3) + x(-5)$

For $t = 0$ $y(0) = x(2) + x(-4)$

For $t = 1$ $y(1) = x(1) + x(-3)$

For some values of t, the output depends on the future input. Therefore, the system is non-causal.

(c) Given $y(t) = \int_{-\infty}^{3t} x(\tau)\,d\tau$

For $t = 0$ $y(0) = \int_{-\infty}^{0} x(\tau)\,d\tau = p(0) - p(-\infty)$

For $t = 1$ $y(1) = \int_{-\infty}^{3} x(\tau)\,d\tau = p(3) - p(-\infty)$

where $\int x(\tau)\,d\tau = p(\tau)$

The output $y(1)$ depends on the future value $p(3)$. Therefore, the system is non-causal.

(d) Given $y(t) = x(t/2)$

For $t = -2$ $y(-2) = x(-1)$

For $t = 0$ $y(0) = x(0)$

For $t = 2$ $y(2) = x(1)$

For negative values of t, the output depends on the future input. Therefore, the system is non-causal.

(e) Given

$$y(t) = x[\sin 2t]$$
$$y(\pi) = x[\sin 2\pi] = x(0)$$
$$y(-\pi) = x[\sin(-2\pi)] = x(0)$$

As the output at $(-\pi)$ depends on the input that occurs later, the system is non-causal.

(f) Given

$$y(n) = x(n) + x(n-2)$$

For $n = -2$ $\quad y(-2) = x(-2) + x(-4)$

For $n = 0$ $\quad y(0) = x(0) + x(-2)$

For $n = 2$ $\quad y(2) = x(2) + x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(g) Given

$$y(n) = x(2n)$$

For $n = -2$ $\quad y(-2) = x(-4)$

For $n = 0$ $\quad y(0) = x(0)$

For $n = 2$ $\quad y(2) = x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(h) Given

$$y(n) = \sin[x(n)]$$

For $n = -2$ $\quad y(-2) = \sin[x(-2)]$

For $n = 0$ $\quad y(0) = \sin[x(0)]$

For $n = 2$ $\quad y(2) = \sin[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(i) Given

$$y(n) = x(-n)$$

For $n = -2$ $\quad y(-2) = x(2)$

For $n = 0$ $\quad y(0) = x(0)$

For $n = 2$ $\quad y(2) = x(-2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

2.2.4 Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system, and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(t)$ for an input $x(t)$ must produce an output $ay(t)$ for an input $ax(t)$.

Superposition property means a system which produces an output $y_1(t)$ for an input $x_1(t)$ and an output $y_2(t)$ for an input $x_2(t)$ must produce an output $y_1(t) + y_2(t)$ for an input $x_1(t) + x_2(t)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(t)$ produces an output $y_1(t)$ and an arbitrary input $x_2(t)$ produces an output $y_2(t)$, then the weighted sum of inputs $ax_1(t) + bx_2(t)$, where a and b are constants, produces an output $ay_1(t) + by_2(t)$ which is the sum of weighted outputs. That is,

$$T[ax_1(t) + bx_2(t)] = aT[x_1(t)] + bT[x_2(t)]$$

For discrete-time linear system,

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its derivative or a constant, the system will definitely be non-linear.

Few examples of linear systems are filters, communication channels, etc.

EXAMPLE 2.3 Check whether the following systems are linear or not:

(a) $\dfrac{d^2y(t)}{dt^2} + 2t\,y(t) = t^2x(t)$ (b) $2\dfrac{dy(t)}{dt} + 5y(t) = x^2(t)$

(c) $\dfrac{dy(t)}{dt} + y(t) = x(t)\dfrac{dx(t)}{dt}$ (d) $y(t) = x(t^2)$

(e) $y(t) = \displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau$ (f) $y(t) = 2x^2(t)$

(g) $y(t) = e^{x(t)}$ (h) $y(n) = n^2x(n)$

(i) $y(n) = x(n) + \dfrac{1}{2x(n-2)}$ (j) $y(n) = 2x(n) + 4$

(k) $y(n) = x(n)\cos\omega n$

Solution:

(a) Given $$\frac{d^2y(t)}{dt^2} + 2t\,y(t) = t^2x(t)$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$\frac{d^2y_1(t)}{dt^2} + 2ty_1(t) = t^2x_1(t)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$\frac{d^2y_2(t)}{dt^2}+2ty_2(t)=t^2x_2(t)$$

Linear combination of the above equations gives

$$a\frac{d^2y_1(t)}{dt^2}+a2ty_1(t)+b\frac{d^2y_2(t)}{dt^2}+b2ty_2(t)=at^2x_1(t)+bt^2x_2(t)$$

i.e. $$\frac{d^2}{dt^2}\underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}}+2t\underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}}=t^2\underbrace{[ax_1(t)+bx_2(t)]}_{\text{Weighted sum of inputs}}$$

This shows that the weighted sum of inputs to the system produces an output which is equal to the weighted sum of outputs to each of the individual inputs. Therefore, the system is linear.

Note: If RHS is function of weighted sum of inputs and LHS is function of weighted sum of outputs, then the system is linear.

(b) Given $$2\frac{dy(t)}{dt}+5y(t)=x^2(t)$$

If an input $x_1(t)$ produces an output $y_1(t)$, then

$$2\frac{dy_1(t)}{dt}+5y_1(t)=x_1^2(t)$$

Similarly, if an input $x_2(t)$ produces an output $y_2(t)$, then

$$2\frac{dy_2(t)}{dt}+5y_2(t)=x_2^2(t)$$

The linear combination of the above equations can be written as:

$$a2\frac{dy_1(t)}{dt}+a5y_1(t)+b2\frac{dy_2(t)}{dt}+b5y_2(t)=ax_1^2(t)+bx_2^2(t)$$

i.e. $$2\frac{d}{dt}\underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}}+5\underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}}=\underbrace{ax_1^2(t)+bx_2^2(t)}_{\text{Not a weighted sum of inputs}}$$

The RHS is not a function of weighted sum of inputs. Hence superposition principle is not satisfied. Therefore, the system is non-linear.

(c) Given $$\frac{dy(t)}{dt}+y(t)=x(t)\frac{dx(t)}{dt}$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$\frac{dy_1(t)}{dt}+y_1(t)=x_1(t)\frac{dx_1(t)}{dt}$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$\frac{dy_2(t)}{dt} + y_2(t) = x_2(t)\frac{dx_2(t)}{dt}$$

The linear combination of the above equations can be written as:

$$a\frac{dy_1(t)}{dt} + ay_1(t) + b\frac{dy_2(t)}{dt} + by_2(t) = ax_1(t)\frac{dx_1(t)}{dt} + bx_2(t)\frac{dx_2(t)}{dt}$$

i.e. $$\frac{d}{dt}\underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}} + \underbrace{[ay_1(t)+by_2(t)]}_{\text{Weighted sum of outputs}} = \underbrace{\left[ax_1(t)\frac{dx_1(t)}{dt} + bx_2(t)\frac{dx_2(t)}{dt}\right]}_{\text{Not a weighted sum of inputs}}$$

Since the RHS is not a weighted sum of inputs, superposition principle is not satisfied. Therefore, the system is non-linear.

(d) Given $$y(t) = x(t^2)$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$y_1(t) = x_1(t^2)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$y_2(t) = x_2(t^2)$$

The linear combination of the above equations can be written as:

$$\underbrace{ay_1(t) + by_2(t)}_{\text{Weighted sum of outputs}} = \underbrace{ax_1(t^2) + bx_2(t^2)}_{\text{Weighted sum of inputs}}$$

The LHS is a function of weighted sum of outputs and the RHS is a function of weighted sum of inputs. The superposition principle is satisfied. Therefore, the system is linear.

(e) Given $$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$y(t) = T[x(t)] = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$y_1(t) = T[x_1(t)] = \int_{-\infty}^{t} x_1(\tau)\,d\tau$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$y_2(t) = T[x_2(t)] = \int_{-\infty}^{t} x_2(\tau)\,d\tau$$

The weighted sum of outputs is:

$$ay_1(t)+by_2(t)=a\int_{-\infty}^{t} x_1(\tau)\,d\tau+b\int_{-\infty}^{t} x_2(\tau)\,d\tau=\int_{-\infty}^{t}[ax_1(\tau)+bx_2(\tau)]d\tau$$

The output due to weighted sum of inputs is:

$$y_3(t)=T[ax_1(t)+bx_2(t)]=\int_{-\infty}^{t}[ax_1(\tau)+bx_2(\tau)]\,d\tau$$

$$y_3(t)=ay_1(t)+by_2(t)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the system is linear.

(f) Given

$$y(t)=2x^2(t)$$

$$y(t)=T[x(t)]=2x^2(t)$$

For an input $x_1(t)$,

$$y_1(t)=T[x_1(t)]=2x_1^2(t)$$

For an input $x_2(t)$,

$$y_2(t)=T[x_2(t)]=2x_2^2(t)$$

The weighted sum of outputs is:

$$ay_1(t)+by_2(t)=a[2x_1^2(t)]+b[2x_2^2(t)]=2[ax_1^2(t)+bx_2^2(t)]$$

The output due to weighted sum of inputs is:

$$y_3(t)=T[ax_1(t)+bx_2(t)]=2[ax_1(t)+bx_2(t)]^2$$

$$y_3(t)\neq ay_1(t)+by_2(t)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the system is non-linear.

(g) Given

$$y(t)=e^{x(t)}$$

$$y(t)=T[x(t)]=e^{x(t)}$$

For an input $x_1(t)$,

$$y_1(t)=T[x_1(t)]=e^{x_1(t)}$$

For an input $x_2(t)$,

$$y_2(t)=T[x_2(t)]=e^{x_2(t)}$$

The weighted sum of outputs is:

$$ay_1(t)+by_2(t)=ae^{x_1(t)}+be^{x_2(t)}$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = e^{[ax_1(t)+bx_2(t)]}$$
$$y_3(t) \neq ay_1(t) + by_2(t)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore the system is non-linear.

(h) Given

$$y(n) = n^2x(n)$$
$$y(n) = T[x(n)] = n^2x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$$\therefore \quad y_1(n) = T[x_1(n)] = n^2x_1(n)$$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$$\therefore \quad y_2(n) = T[x_2(n)] = n^2x_2(n)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2x_1(n)] + b[n^2x_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2[ax_1(n) + bx_2(n)]$$
$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(i) Given

$$y(n) = x(n) + \frac{1}{2x(n-2)}$$
$$y(n) = T[x(n)] = x(n) + \frac{1}{2x(n-2)}$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{2x_1(n-2)}$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{2x_2(n-2)}$$

The weighted sum of outputs is given by

$$ay_1(n) + by_2(n) = a\left[x_1(n) + \frac{1}{2x_1(n-2)}\right] + b\left[x_2(n) + \frac{1}{2x_2(n-2)}\right]$$
$$= [ax_1(n) + bx_2(n)] + \frac{a}{2x_1(n-2)} + \frac{b}{2x_2(n-2)}$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n)+bx_2(n)] = [ax_1(n)+bx_2(n)] + \frac{1}{2[ax_1(n-2)+bx_2(n-2)]}$$

$$y_3(n) \neq ay_1(n)+by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(j) Given

$$y(n) = 2x(n)+4$$

$$y(n) = T[x(n)] = 2x(n)+4$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = 2x_1(n)+4$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = 2x_2(n)+4$$

The weighted sum of outputs is:

$$ay_1(n)+by_2(n) = a[2x_1(n)+4]+b[2x_2(n)+4] = 2[ax_1(n)+bx_2(n)]+4(a+b)$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n)+bx_2(n)] = 2[ax_1(n)+bx_2(n)]+4$$

$$y_3(n) \neq ay_1(n)+by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(k) Given

$$y(n) = x(n)\cos\omega n$$

$$y(n) = T[x(n)] = x(n)\cos\omega n$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n)\cos\omega n$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n)\cos\omega n$$

The weighted sum of outputs is:

$$ay_1(n)+by_2(n) = ax_1(n)\cos\omega n + bx_2(n)\cos\omega n = [ax_1(n)+bx_2(n)]\cos\omega n$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n)+bx_2(n)] = [ax_1(n)+bx_2(n)]\cos\omega n$$

$$y_3(n) = ay_1(n)+by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

2.2.5 Time-invariant and Time-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied.

A system is said to be time-invariant (or shift-invariant) if its input/output characteristics do not change with time, i.e. if a time shift in the input results in a corresponding time shift in the output as shown in Figure 2.4, i.e.

If
$$x(t) \rightarrow y(t)$$

Then
$$x(t-T) \rightarrow y(t-T)$$

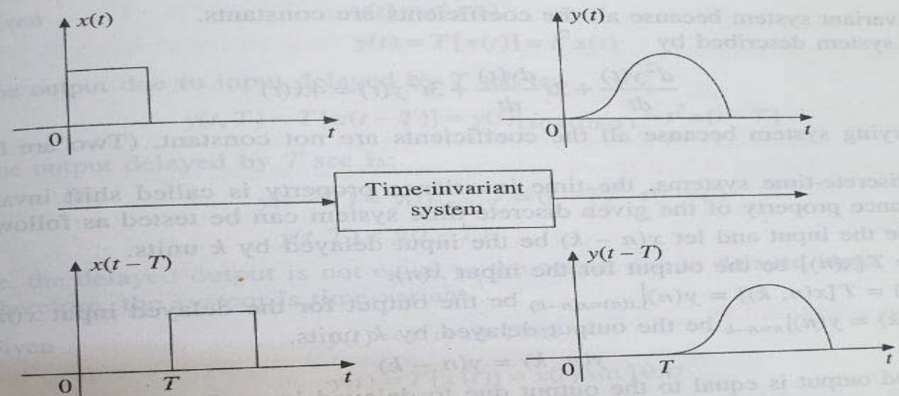

Figure 2.4 Time-invariant system.

A system not satisfying the above requirements is called a time-varying system (or shift varying system)

The time-invariance property of the given continuous-time system can be tested as follows:

Let $x(t)$ be the input and let $x(t-T)$ be the input delayed by T units.

$y(t) = T[x(t)]$ be the output for an input $x(t)$.

$y(t, T) = T[x(t-T)] = y(t)|_{x(t)=x(t-T)}$ be the output for the delayed input $x(t-T)$.

$y(t-T) = y(t)|_{t=t-T}$ be the output delayed by T units.

If
$$y(t, T) = y(t-T)$$

i.e. if the delayed output is equal to the output due to delayed input for all possible values of t, then the system is time-invariant.

On the other hand, if

$$y(t, T) \neq y(t - T)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the continuous-time system is described by differential equation, the time-invariance can be found by observing the coefficients of the differential equation. If the coefficients of the differential equation are constants, then the system is time-invariant. If the coefficients are functions of time, then the system is time-variant.

The system described by

$$3\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 5x(t)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$\frac{d^2y(t)}{dt} + 2t\frac{dy(t)}{dt} + 3t^2y(t) = 4x(t)$$

is time varying system because all the coefficients are not constant. (Two are functions of time).

For discrete-time systems, the time invariance property is called shift invariance. The time invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n-k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

$y(n, k) = T[x(n, k)] = y(n)|_{x(n)=x(n-k)}$ be the output for the delayed input $x(n-k)$.

$y(n-k) = y(n)|_{n=n-k}$ be the output delayed by k units.

If

$$y(n, k) = y(n - k)$$

i.e. if delayed output is equal to the output due to delayed input for all possible values of k, then the system is time-invariant.

On the other hand, if

$$y(n, k) \neq y(n - k)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the discrete-time system is described by difference equation, the time invariance can be found by observing the coefficients of the difference equation. If the coefficients of the difference equation are constants, then the system is time-invariant. If the coefficients are functions of time, then the system is time-variant.

The system described by

$$y(n) + 3y(n-1) + 5y(n-2) = 2x(n)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$y(n) - 2ny(n-1) + 3n^2y(n-2) = x(n) + x(n-1)$$

is time-varying system because all the coefficients are not constant (Two are functions of time).

The systems satisfying both linearity and time-invariance properties are popularly known as linear time-invariant or simply LTI systems.

EXAMPLE 2.4 Determine whether the following systems are time-invariant or not:

(a) $y(t) = t^2 x(t)$ (b) $y(t) = x(t)\sin 10\pi t$

(c) $y(t) = x(t^2)$ (d) $y(t) = x(-2t)$

(e) $y(t) = e^{2x(t)}$ (f) $y(n) = x(n/2)$

(g) $y(n) = x(n)$ (h) $y(n) = x^2(n-2)$

(i) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given

$$y(t) = t^2 x(t)$$

$$y(t) = T[x(t)] = t^2 x(t)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = t^2 x(t-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = (t-T)^2\, x(t-T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given

$$y(t) = x(t)\sin 10\pi t$$

$$y(t) = T[x(t)] = x(t)\sin 10\pi t$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(t-T)\sin 10\pi t$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x(t-T)\sin 10\pi(t-T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(c) Given

$$y(t) = x(t^2)$$

$$y(t) = T[x(t)] = x(t^2)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(t^2 - T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x[(t-T)^2]$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input.
Therefore, the system is time-variant.

(d) Given $\quad y(t) = x(-2t)$

$$y(t) = T[x(t)] = x(-2t)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(-2t-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x[-2(t-T)] = x(-2t+2T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input.
Therefore, the system is time-variant.

(e) Given $\quad y(t) = e^{2x(t)}$

$$y(t) = T[x(t)] = e^{2x(t)}$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = e^{2x(t-T)}$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = e^{2x(t-T)}$$

$$y(t, T) = y(t-T)$$

i.e. the delayed output is equal to the output due to delayed input.
Therefore, the system is time-invariant.

(f) Given $\quad y(n) = x\left(\frac{n}{2}\right)$

$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2}-k\right)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n,k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(g) Given

$$y(n) = x(n)$$

$$y(n) = T[x(n)] = x(n)$$

The output due to input delayed by k units is:

$$y(n,k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)$$

$$y(n,k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(h) Given

$$y(n) = x^2(n-2)$$

$$y(n) = T[x(n)] = x^2(n-2)$$

The output due to input delayed by k units is:

$$y(n,k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-2-k)$$

$$y(n,k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(i) Given

$$y(n) = x(n) + nx(n-2)$$

$$y(n) = T[x(n)] = x(n) + nx(n-2)$$

The output due to input delayed by k units is:

$$y(n,k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k) + nx(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k) + (n-k)\,x(n-k-2)$$

$$y(n,k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

EXAMPLE 2.5 Show that the following systems are linear time-invariant systems:

(a) $y(t) = x\left(\frac{t}{2}\right)$ (b) $y(t) = \begin{cases} x(t) + x(t-2) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

Solution: To show that a given system is a linear time-invariant system we have to show separately that it is linear and time-invariant.

(a) Given $$y(t) = x\left(\frac{t}{2}\right)$$

For inputs $x_1(t)$ and $x_2(t)$,

$$y_1(t) = x_1\left(\frac{t}{2}\right)$$

$$y_2(t) = x_2\left(\frac{t}{2}\right)$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = ax_1\left(\frac{t}{2}\right) + bx_2\left(\frac{t}{2}\right)$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = ax_1\left(\frac{t}{2}\right) + bx_2\left(\frac{t}{2}\right)$$

$$y_3(t) = ay_1(t) + by_2(t)$$

So the system is linear.

$$y(t,T) = y(t)\big|_{x(t)=x(t-T)} = x\left(\frac{t}{2} - T\right)$$

$$y(t-T) = y(t)\big|_{t=t-T} = x\left(\frac{t-T}{2}\right)$$

$$y(t,T) \neq y(t-T)$$

So the system is time varying.

Hence the given system is linear but time varying. It is not a linear time-invariant system.

(b) Given $$y(t) = \begin{cases} x(t) + x(t-2) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

For inputs $x_1(t)$ and $x_2(t)$,

$$y_1(t) = x_1(t) + x_1(t-2) \quad \text{for } t \geq 0$$

$$y_2(t) = x_2(t) + x_2(t-2) \quad \text{for } t \geq 0$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = a[x_1(t) + x_1(t-2)] + b[x_2(t) + x_2(t-2)]$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = [ax_1(t) + bx_2(t)] + ax_1(t-2) + bx_2(t-2)$$

$$y_3(t) = ay_1(t) + by_2(t)$$

So the system is linear.

$$y(t, T) = y(t)\big|_{x(t)=x(t-T)} = x(t-T) + x(t-2-T)$$

$$y(t-T) = y(t)\big|_{t=t-T} = x(t-T) + x(t-T-2)$$

$$y(t, T) = y(t-T)$$

So the system is time-invariant. Hence the given system is linear time-invariant.

EXAMPLE 2.6 Check whether the following systems are:

1. Static or dynamic
2. Linear or non-linear
3. Causal or non-causal
4. Time-invariant or time-variant

(a) $\dfrac{d^3y(t)}{dt^3} + 2\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3y^2(t) = x(t+1)$

(b) $\dfrac{d^2y(t)}{dt^2} + 2y(t)\dfrac{dy(t)}{dt} + 3t\,y(t) = x(t)$

(c) $y(t) = ev\{x(t)\}$ (d) $y(t) = at^2x(t) + bt\,x(t-4)$

(e) $y(n) = x(n)\,x(n-2)$ (f) $y(n) = \log_{10}|x(n)|$

(g) $y(n) = a^n u(n)$ (h) $y(n) = x^2(n) + \dfrac{1}{x^2(n-1)}$

Solution:

(a) Given $$\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y^2(t) = x(t+1)$$

1. The system is described by a differential equation. Hence the system is dynamic.
2. There is a square term of output [i.e. $y^2(t)$]. So the system is non-linear. This can be proved.

   Let an input $x_1(t)$ produce an output $y_1(t)$. So the differential equation becomes

$$\frac{d^3y_1(t)}{dt^3} + 2\frac{d^2y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} + 3y_1^2(t) = x_1(t+1)$$