Let an input $x_2(t)$ produce an output $y_2(t)$. So the differential equation becomes

$$\frac{d^3y_2(t)}{dt^3}+2\frac{d^2y_2(t)}{dt^2}+4\frac{dy_2(t)}{dt}+3y_2^2(t)=x_2(t+1)$$

The linear combination of the above equations becomes

$$a\frac{d^3y_1(t)}{dt^3}+a2\frac{d^2y_1(t)}{dt^2}+a4\frac{dy_1(t)}{dt}+a3y_1^2(t)$$

$$+b\frac{d^3y_2(t)}{dt^3}+b2\frac{d^2y_2(t)}{dt^2}+b4\frac{dy_2(t)}{dt}+b3y_2^2(t)$$

$$=ax_1(t+1)+bx_2(t+1)$$

i.e. $$\frac{d^3}{dt^3}[ay_1(t)+by_2(t)]+2\frac{d^2}{dt^2}[ay_1(t)+by_2(t)]+4\frac{d}{dt}[ay_1(t)+by_2(t)]$$

$$+\underbrace{3[ay_1^2(t)+by_2^2(t)]}_{\text{Not a weighted sum of outputs}}=ax_1(t+1)+bx_2(t+1)$$

Since one term in LHS is not a weighted sum of outputs, the superposition principle is not valid. Hence the system is non-linear.

3. The output depends on future values of input. Hence the system is non-causal.
4. All the coefficients of the differential equation are constant. Hence the system is time-invariant.

So the given system is dynamic, non-linear, non-causal and time-invariant.

(b) Given $$\frac{d^2y(t)}{dt^2}+2y(t)\frac{dy(t)}{dt}+3t\,y(t)=x(t)$$

1. The system is described by a differential equation. Hence the system is dynamic.
2. There is a term with product of output and its derivative {i.e. $y(t)[dy(t)/dt]$}. Hence the system is non-linear. This can be proved.

Let an input $x_1(t)$ produce an output $y_1(t)$

Then $$\frac{d^2y_1(t)}{dt^2}+2y_1(t)\frac{dy_1(t)}{dt}+3t\,y_1(t)=x_1(t)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$\frac{d^2y_2(t)}{dt^2}+2y_2(t)\frac{dy_2(t)}{dt}+3ty_2(t)=x_2(t)$$

The linear combination of the above equations gives

$$a\frac{d^2y_1(t)}{dt^2}+a2y_1(t)\frac{dy_1(t)}{dt}+a3ty_1(t)+\frac{bd^2y_2(t)}{dt^2}+b2y_2(t)\frac{dy_2(t)}{dt^2}+b3t\,y_2(t)$$

$$=ax_1(t)+bx_2(t)$$

i.e. $$\frac{d^2}{dt^2}[ay_1(t)+by_2(t)]+2\underbrace{\left[ay_1(t)\frac{dy_1(t)}{dt}+by_2(t)\frac{dy_2(t)}{dt}\right]}_{\text{Not a weighted sum of outputs}}+3t[ay_1(t)+by_2(t)]$$

$$=[ax_1(t)+bx_2(t)]$$

Since one term in LHS is not a weighted sum of outputs, the superposition principle is not valid. Hence the system is non-linear.

3. The output depends on present input only. Hence the system is causal.
4. All the coefficients of the differential equation are not constants. One coefficient is a function of time. Hence the system is time-variant.

So the given system is dynamic, non-linear, causal and time-variant.

(c) Given $$y(t)=ev\{x(t)\}$$

$$y(t)=ev\{x(t)\}=\frac{1}{2}[x(t)+x(-t)]$$

1. For positive values of t, the output depends on past values of input and for negative values of t, the output depends on future values of input. Hence the system is dynamic.
2. $$y(t)=T[x(t)]=\frac{1}{2}[x(t)+x(-t)]$$

For an input $x_1(t)$,

$$y_1(t)=\frac{1}{2}[x_1(t)+x_1(-t)]$$

For an input $x_2(t)$,

$$y_2(t)=\frac{1}{2}[x_2(t)+x_2(-t)]$$

The weighted sum of outputs is:

$$ay_1(t)+by_2(t)=a\frac{1}{2}[x_1(t)+x_1(-t)]+b\frac{1}{2}[x_2(t)+x_2(-t)]$$

$$=\frac{1}{2}\{[ax_1(t)+bx_2(t)]+[ax_1(-t)+bx_2(-t)]\}$$

The output due to weighted sum of inputs is:

$$y_3(t)=T[ax_1(t)+bx_2(t)]=\frac{1}{2}\{[ax_1(t)+bx_2(t)]+[ax_1(-t)+bx_2(-t)]\}$$

$$y_3(t)=ay_1(t)+by_2(t)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. Hence superposition principle is valid and the system is linear.

3. $$y(-2) = \frac{1}{2}[x(-2) + x(2)]$$

i.e. for negative values of t, the output depends on future values of input. Hence the system is non-causal.

4. Given $$y(t) = \frac{1}{2}[x(t) + x(-t)]$$

The output due to input delayed by T units is:

$$y(t, T) = T[x(t-T)] = y(t)|_{x(t)=x(t-T)} = \frac{1}{2}[x(t-T) + x(-t-T)]$$

The output delayed by T units is:

$$y(t-T) = y(t)|_{t=t-T} = \frac{1}{2}[x(t-T) + x(-t+T)]$$

$$y(t, T) \neq y(t-T)$$

So the system is time-variant.

So the given system is dynamic, linear, non-causal and time-variant.

(d) Given $$y(t) = at^2 x(t) + bt\, x(t-4)$$

1. The output depends on past inputs. So it requires memory. Hence it is a dynamic system.
2. Given $$y(t) = at^2 x(t) + bt\, x(t-4)$$

   For an input $x_1(t)$,

   $$y_1(t) = at^2 x_1(t) + btx_1(t-4)$$

   For an input $x_2(t)$,

   $$y_2(t) = at^2 x_2(t) + btx_2(t-4)$$

   The weighted sum of outputs is:

   $$py_1(t) + qy_2(t) = pat^2 x_1(t) + pbt\, x_1(t-4) + qat^2 x_2(t) + qbt\, x_2(t-4)]$$
   $$= at^2[px_1(t) + qx_2(t)] + bt[px_1(t-4) + qx_2(t-4)]$$

   The output due to weighted sum of inputs is:

   $$y_3(t) = T[px_1(t) + qx_2(t)] = at^2[px_1(t) + qx_2(t)] + bt[px_1(t-4) + qx_2(t-4)]$$
   $$y_3(t) = ay_1(t) + by_2(t)$$

   Superposition principle is satisfied. Hence the system is linear.
3. The output depends only on the present and past inputs. It does not depend on future inputs. Hence the system is causal.

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2 + \frac{1}{[ax_1(n-1) + bx_2(n-1)]^2}$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

Hence the system is non-linear.

3. The output does not depend on future values of input. Hence the system is causal.

4. Given $$y(n) = T[x(n)] = x^2(n) + \frac{1}{x^2(n-1)}$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-k) + \frac{1}{x^2(n-1-k)}$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-k) + \frac{1}{x^2(n-k-1)}$$

$$y(n, k) = y(n-k)$$

Hence the system is time-invariant.

So the given system is dynamic, non-linear, causal and time-invariant.

2.2.6 Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value. For example, a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of a stable system does not diverge or does not grow unreasonably large.

Let the input signal $x(t)$ be bounded (finite), i.e.

$$|x(t)| \leq M_x < \infty \text{ for all } t$$

where M_x is a positive real number.

If

$$|y(t)| \leq M_y < \infty$$

i.e. if $y(t)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely integrable for a continuous-time system or absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a system to be BIBO stable is given by the expression

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

where $h(t)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(t)$ as input and $y(t)$ as output. The input and output of the system are related by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking absolute values on both sides, we have

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

Using the fact that the absolute value of the integral of the product of two terms is always less than or equal to the integral of the product of their absolute values, we have

$$\left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\, d\tau$$

If the input $x(\tau)$ is bounded, i.e. there exists a finite number M_x such that,

$$|x(\tau)| \le M_x < \infty$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau$$

Changing the variables by $m = t - \tau$, the output is bounded if

$$\int_{-\infty}^{\infty} |h(m)|\, dm < \infty$$

Replacing m by t, we have $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

which is the necessary and sufficient condition for a system to be BIBO stable.

Figure 2.5 shows bounded and unbounded signals.

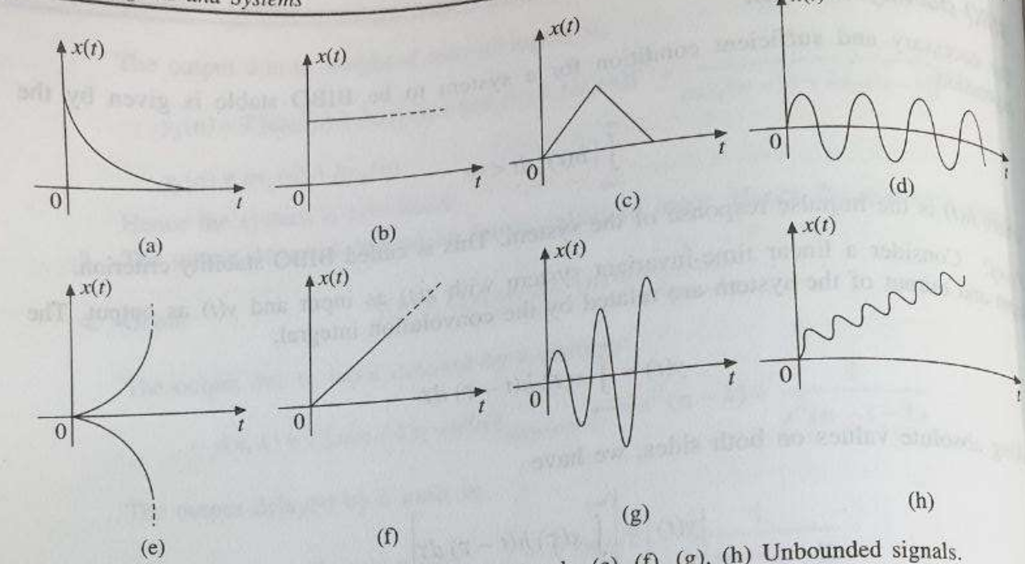

Figure 2.5 (a), (b), (c), (d) Bounded signals, (e), (f), (g), (h) Unbounded signals.

Figure 2.6 shows stable and unstable systems.

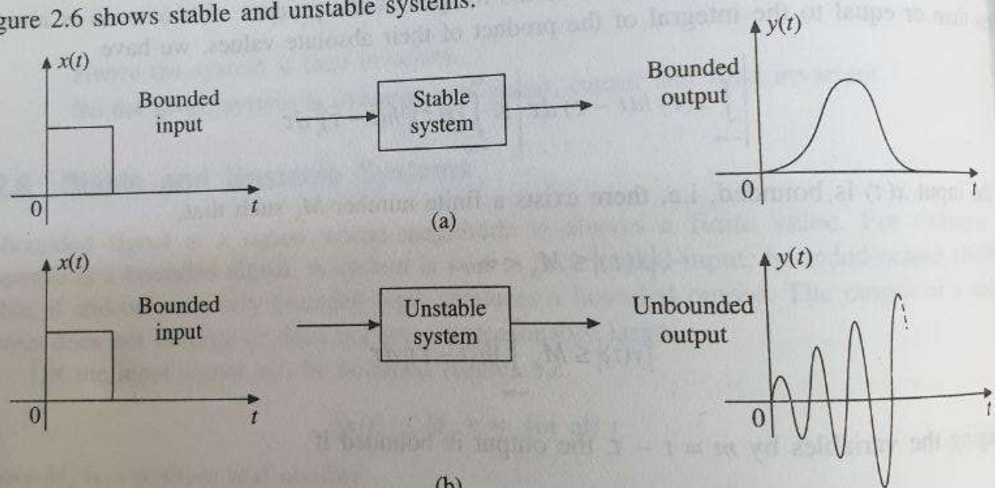

Figure 2.6 (a) Stable system, (b) Unstable system.

EXAMPLE 2.7 Find whether the following systems are stable or not:

(a) $y(t) = e^{x(t)};\ |x(t)| \le 8$

(b) $y(t) = (t+5)\,u(t)$

(c) $h(t) = (2 + e^{-3t})\,u(t)$

(d) $h(t) = e^{2t}\,u(t)$

(e) $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(f) $h(t) = \frac{1}{RC}\,e^{-t/RC}\,u(t)$

(g) $h(t) = \omega_0 \left|\sin \omega_0 t\, u(t)\right|$

Solution:

(a) Given
$$y(t) = e^{x(t)};\ |x(t)| \le 8$$

Here the input is bounded, $|x(t)| \le 8$

Therefore for stability, the output must be bounded.

The output $y(t)$ becomes
$$e^{-8} \le y(t) \le e^{8}$$

Hence $y(t)$ is also bounded. Therefore, the system is stable.

(b) Given
$$y(t) = (t+5)\,u(t)$$

$\therefore$
$$y(t) = t + 5 \quad \text{for } t \ge 0$$

So, as $t \to \infty$, $y(t) \to \infty$

Hence the output grows without any bound and hence the given system is unstable.

(c) Given
$$h(t) = (2 + e^{-3t})\,u(t)$$

For a system to be stable,
$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Here
$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} (2 + e^{-3t})\,u(t)\,dt = \int_{0}^{\infty} (2 + e^{-3t})\,dt = \left[2t + \frac{e^{-3t}}{-3}\right]_0^{\infty} = \infty$$

Since the impulse response is not absolutely integrable, i.e. since $\int_{-\infty}^{\infty} |h(t)|\,dt = \infty$, the system is unstable.

(d) Given
$$h(t) = e^{2t}u(t)$$

$\therefore$
$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \left|e^{2t}u(t)\right|\,dt = \int_{0}^{\infty} e^{2t}\,dt = \left[\frac{e^{2t}}{2}\right]_0^{\infty} = \infty$$

Since the impulse response is not absolutely integrable, i.e. since $\int_{-\infty}^{\infty} |h(t)|\,dt = \infty$, the system is unstable.

(e) Given
$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

For an input $\delta(t)$,
$$y(t) = h(t)$$

Therefore,
$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

For stability,
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

In this case,
$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} u(t)\, dt = \infty$$

Therefore, the impulse response is unbounded and the system is unstable.

(f) Given
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

For stability,
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

In this case,
$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| \frac{1}{RC} e^{-t/RC} u(t) \right| dt = \int_{0}^{\infty} \left| \frac{1}{RC} e^{-t/RC} \right| dt = 1$$

Therefore, the system is stable.

(g) Given
$$h(t) = \omega_0 |\sin \omega_0 t\, u(t)|$$
$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \omega_0 \int_{0}^{\infty} |\sin \omega_0 \tau|\, d\tau$$

This integral does not converge and so output is not bounded. Hence the system is unstable.

EXAMPLE 2.8 Check the stability of the system defined by

(a) $y(n) = ax(n-7)$

(b) $y(n) = x(n) + \frac{1}{2} x(n-1) + \frac{1}{4} x(n-2)$

(c) $h(n) = a^n$ for $0 < n < 11$

(d) $h(n) = 2^n u(n)$

(e) $h(n) = u(n)$

Solution:

(a) Given
$$y(n) = ax(n-7)$$

Let
$$x(n) = \delta(n)$$

Then
$$y(n) = h(n)$$

$\therefore$
$$h(n) = a\delta(n-7)$$

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e. $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a \quad \text{for } n = 7$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$$\therefore \quad h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$, and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-1) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$$\therefore \quad \sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \quad \text{(a finite value)}$$

Hence the system is stable.

(c) Given $$h(n) = a^n \quad \text{for } 0 < n < 11$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \sum_{n=0}^{11} a^n = \frac{1-a^{12}}{1-a}$$

This value is finite for finite value of a. Hence the system is stable if a is finite.

(d) Given $$h(n) = 2^n\, u(n)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = \infty$$

The impulse response is not absolutely summable. Hence this system is unstable.

(e) Given $$h(n) = u(n)$$

For stability,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

So the output is not bounded and the system is unstable.

EXAMPLE 2.9 Check whether the following digital systems are BIBO stable or not:

(a) $y(n) = ax(n+1) + bx(n-1)$

(b) $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

(c) $y(n) = ax(n) + b$

(d) $y(n) = e^{-x(n)}$

(e) $y(n) = ax(n) + bx^2(n-1)$

Solution:

(a) Given $$y(n) = ax(n+1) + bx(n-1)$$

If $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

Hence the impulse response is:

$$h(n) = a\delta(n+1) + b\delta(n-1)$$

When $n = 0, h(0) = a\delta(1) + b\delta(-1) = 0$

When $n = 1, h(1) = a\delta(2) + b\delta(0) = b$

When $n = 2, h(2) = a\delta(3) + b\delta(1) = 0$

In general,

$$h(n) = \begin{cases} b & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} |h(n)| = b$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So the system is BIBO stable if $|b| < \infty$.

(b) Given $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

$$h(n) = \text{maximum of } [\delta(n), \delta(n-1), \delta(n-2)]$$

$\therefore$

$$h(0) = \text{maximum of } [\delta(0), \delta(-1), \delta(-2)] = 1$$
$$h(1) = \text{maximum of } [\delta(1), \delta(0), \delta(-1)] = 1$$
$$h(2) = \text{maximum of } [\delta(2), \delta(1), \delta(0)] = 1$$
$$h(3) = \text{maximum of } [\delta(3), \delta(2), \delta(1)] = 0$$

Similarly $h(4) = 0 = h(5) = h(6) \ldots$

$\therefore$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots$$
$$= 1 + 1 + 1 + 0 + 0 + \cdots = 3$$

So the given system is BIBO stable.

(c) Given $y(n) = ax(n) + b$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = a\delta(n) + b$$

When $n = 0,\ h(0) = a\delta(0) + b = a + b$

When $n = 1,\ h(1) = a\delta(1) + b = b$

Here, $h(1) = h(2) = \ldots = h(n) = b$

Therefore,

$$h(n) = \begin{cases} a+b & \text{when } n = 0 \\ b & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots + \cdots$$
$$= |a+b| + |b| + |b| + \cdots + |b| + \cdots$$

This series never converges since the ratio between the successive terms is one. Hence the given system is BIBO unstable.

(d) Given $y(n) = e^{-x(n)}$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = e^{-\delta(n)}$$

When $n=0, h(0) = e^{-\delta(0)} = e^{-1}$

When $n=1, h(1) = e^{-\delta(1)} = e^{0} = 1$

In general,

$$h(n) = \begin{cases} e^{-1} & \text{when } n = 0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= e^{-1} + 1 + 1 + 1 + \cdots + 1 + \cdots$$

Since the given sequence never converges, it is BIBO unstable.

(e) Given $y(n) = ax(n) + bx^2(n-1)$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = a\delta(n) + b\delta^2(n-1)$$

When $n=0, h(0) = a\delta(0) + b\delta^2(-1) = a$

When $n=1, h(1) = a\delta(1) + b\delta^2(0) = b$

When $n=2, h(2) = a\delta(2) + b\delta^2(1) = 0$

Hence,

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= |a| + |b| + 0 + 0 + \cdots$$

Hence, the given system is BIBO stable if $|a| + |b| < \infty$.

EXAMPLE 2.10 Determine whether each of the system with impulse response/output listed below is (i) causal (ii) stable.

(a) $h(n) = 3^n u(-n)$

(b) $h(n) = \cos \dfrac{n\pi}{2}$

(c) $h(n) = \delta(n) + \cos n\pi$

(d) $h(n) = e^{3n} u(n-2)$

(e) $y(n) = \cos x(n)$

(f) $y(n) = \sum_{k=-\infty}^{n+5} x(k)$