

Chapter eight

Matrices and Determinants

A matrix is a rectangular array of elements (scalars) from a field. The order, or size, of a matrix is specified by the number of rows and the number of columns, i.e. A an “ m by n ” matrix has m rows and n columns, and the element in the i th row and j th column is often denoted by a_{ij} :

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A vector is a matrix with a single row (or column) of n elements , i.e. the column vector is:-

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \quad \text{and row vector is } A = [a_1 \ a_2 \ \cdot \ \cdot \ a_n]$$

The matrix is square if the number of rows and columns are equal (i.e. $m = n$) and the elements a_{ij} of a square matrix are called the main diagonal.

The identity matrix: $I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ is square matrix

with one in each main diagonal position and zeros else.

The diagonal matrix $D = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n \end{bmatrix}$ has the elements

a_1, a_2, \dots, a_n in its main diagonal position and zeros in all other locations, some of the a_i may be zero but not all.

A $n \times n$ triangular matrix has the pattern:-

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

lower triangular matrix *upper triangular matrix*

The $m \times n$ null matrix:- $O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ has zero in each of

its positions.

Elementary operations with matrices and vectors

1. Equality:- Two $m \times n$ matrices and A and B are said to be equal if: $a_{ij} = b_{ij} \quad \forall$ pairs of i and j .

EX-1 – Find the values of x, y for the following matrix equation:

$$\begin{bmatrix} x - 2y & 0 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & x + y \end{bmatrix}$$

Sol. –

$$\left. \begin{array}{l} x - 2y = 3 \quad \dots(1) \\ x + y = 6 \quad \dots(2) * 2 \end{array} \right\} \Rightarrow \frac{2x + 2y = 12}{3x = 15} \Rightarrow \boxed{x = 5}$$

$$\text{substitution } x = 5 \text{ in (2)} \Rightarrow 5 + y = 6 \Rightarrow \boxed{y = 1}$$

2. Addition:- The sum of two matrices of like dimensions is the matrix of the sum of the corresponding elements. If:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

thus:

- 1) $A + B = B + A$
- 2) $A + (B + C) = (A + B) + C$
- 3) $A - (B - C) = A - B + C$

EX-2- Find $A+B$ and $A-B$ if:-

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

Sol.-

$$A + B = \begin{bmatrix} 2+1 & 1-2 & 3+2 \\ 1+2 & 0+3 & -2-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 3 & 3 & -3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-1 & 1-(-2) & 3-2 \\ 1-2 & 0-(+3) & -2-(-1) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

3. **Multiplication by a scalar:-** The matrix A is multiplied by the scalar C by multiplying each element of A by c :-

$$CA = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}$$

EX-3- Assume $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \end{bmatrix}$, find $3A$.

Sol.-

$$3A = \begin{bmatrix} 3*3 & 3*2 & 3*1 \\ 3*0 & 3*5 & 3*(-1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 0 & 15 & -3 \end{bmatrix}$$

4. **Matrix multiplication:-** For the matrix product AB to be defined it is necessary that the number of columns of A be equal to the number of rows of B . The dimensions of such matrices are said to be conformable. If A is of dimensions $m \times p$ and B is $p \times n$, then the ij th element of the product $C=AB$ is computed as:-

$$C_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

This is the sum of the products of corresponding elements in the i th row of A and j th column of B . The dimensions of AB are of course $m \times n$.

EX-4- Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 5 & 4 \\ -1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ find AB .

Sol.-

$$AB = \begin{bmatrix} 1*6+2(-1)+3*0 & 1*5+2*1+3*2 & 1*4+2(-1)+3*0 \\ -1*6+0(-1)+1*0 & -1*5+0*1+1*2 & -1*4+0(-1)+1*0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 & 2 \\ -6 & -3 & -4 \end{bmatrix}$$

Properties of multiplication:-

- a) $A(B + C) = AB + AC$ *distributive law*
- b) $A(BC) = (AB)C$ *associative law*
- c) $AB \neq BA$ *commutative law does not hold*
- d) $AI = IA = A$

EX-5- Assume $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, verify that $AB \neq BA$.

Sol.-

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 6 & 3 \end{bmatrix} \quad \& \quad BA = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 7 \end{bmatrix}$$

Hence $AB \neq BA$

5. Transpose of matrix:- Let A is any $m \times n$ matrix the transpose of A is $n \times m$ matrix A' formed by interchanging the role of rows and columns.

$$A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

If a matrix is square and equal to its transpose, it is said to be symmetric, then $a_{ij} = a_{ji}$ for all pairs of i and j .

Properties of transpose are:-

- a) $(A + B)' = A' + B'$
- b) $(AB)' = B'A'$
- c) $(A')' = A$

EX-6- Assume $A = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 4 & 3 \\ 2 & 1 & -1 \end{bmatrix}$, show that:-

- 1) A is symmetric matrix
- 2) $(A + B)' = A' + B'$
- 3) $(AB)' = B'A'$

Sol.-

$$1) A' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = A \Rightarrow A \text{ is a symmetric matrix.}$$

$$2) L.H.S. = (A+B)' = \begin{bmatrix} 7 & 1 & 5 \\ 7 & 3 & 7 \\ 7 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix}$$

$$R.H.S. = A' + B' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix} = L.H.S.$$

$$\therefore (A+B)' = A' + B'$$

$$3) L.H.S. = (AB)' = \begin{bmatrix} 32 & 10 & 1 \\ 11 & -2 & -7 \\ 40 & 11 & 12 \end{bmatrix}' = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix}$$

$$R.H.S. = B'A' = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix} = L.H.S.$$

$$\therefore (AB)' = B'A'$$

6. Vector inner product:- The inner product of two vectors with the same number of elements is defined to be the sum of the products of the corresponding elements:-

$$A' B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^n a_i b_i$$

Since the inner product is a scalar, hence $A'B = B'A$. Moreover, the inner product of two vectors may be taken the following term:-

$$A B' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [b_1 \quad b_2 \quad \dots \quad b_n] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \dots & \dots & \dots & \dots \\ a_n b_1 & a_n b_2 & \dots & a_n b_n \end{bmatrix}$$

Which is $n \times n$ matrix.

EX-7- Let $A = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, find $A'B$ and AB'

Sol.-

$$A'B = [5 \quad -2 \quad 1] \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 5 * 2 + (-2) * (-1) + 1 * 3 = 15$$

$$AB' = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} [2 \quad -1 \quad 3] = \begin{bmatrix} 10 & -5 & 15 \\ -4 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix}$$

Determinants

The minor of the element a_{ij} in a matrix A is the determinant of the matrix that remains when the row and column containing a_{ij} are deleted. For example, let:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then the minor of } a_{21} \text{ is } \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ then the minor of } a_{34} \text{ is } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

and so on.

The cofactor of a_{ij} is the determinant A_{ij} that is $(-1)^{i+j}$ times the minor of a_{ij} . Thus:-

$$\text{for matrix } (3 \times 3) \Rightarrow A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{for matrix } (4 \times 4) \Rightarrow A_{31} = (-1)^{3+1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

With each square matrix A we associate a number $\det A$ or $|A|$ or $|a_{ij}|$ called the determinant of A , calculated from the entries of A in the following way:-

$$\text{for } n = 1, A = [a] \Rightarrow |A| = a$$

$$\text{for } n = 2, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{for } n = 3, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

- - -
+ + +

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

The determinant of a square matrix can be calculated from the cofactors of any row or any column.

EX-8- Find the determinant of the matrix:- $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

Sol.-
1st method

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{matrix} 2 & 1 & 3 & 2 & 1 \\ 3 & -1 & -2 & 3 & -1 \\ 2 & 3 & 1 & 2 & 3 \end{matrix}$$

- - -
+ + +

$$= 2(-1) \cdot 1 + 1(-2) \cdot 2 + 3 \cdot 3 \cdot 3 - (3(-1) \cdot 2 + 2(-2) \cdot 3 + 1 \cdot 3 \cdot 1)$$

$$= 36$$

2nd method

If we were to expand the determinant by cofactors according to elements of its third column, say, we would get:-

$$\begin{aligned} A &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\ &= 3(-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 3(9 - (-2)) + 2(6 - 2) + (-2 - 3) = 36 \end{aligned}$$

Useful facts about determinants:-

F-1: If two rows of matrix are identical, the determinant is zero.

EX-9 Show that:- $\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$

Sol.-

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix} \begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} \begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} = -18 - 15 - 4 - (-18 - 15 - 4) = 0$$

F-2: Interchanging two rows of matrix changes the sign of its determinants.

EX-10 Show that:- $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = 0 + 3 + 10 - (0 - 12 - 4) = 29$$

$$R.H.S. = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} = -(-4 + 0 - 12 - (3 + 10 + 0)) = 29 = L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$$

F-3: The determinant of the transpose of a matrix is equal to the original determinant.

EX-11 Show that:-
$$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$$

Sol.-

L.H.S. = 29 from ex-10

R.H.S. =
$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix} = 2(-12-4) - 1(4-10) + 5(-2-3) = 0 + 10 + 3 - (0 - 12 - 4) = 29 = \text{L.H.S.}$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$$

F-4: If each element of same row (or column) of a matrix is multiplied by a constant C , the determinant is multiplied by C .

EX-12 Show that:-
$$\begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3 \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

Sol.-

L.H.S. =
$$\begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 6(0-36) - 3(4-12) + 15(-2-3) = 0 + 9 + 30 - (0 - 36 - 12) = 87$$

R.H.S. = $3 * 29 = 87 = \text{L.H.S.}$

$$\therefore \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3 \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

F-5: If all elements of a matrix above the main diagonal (or all below it) are zero, the determinant of the matrix is the product of the elements on the main diagonal.

EX-13 Find:-
$$\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}$$

Sol.-

$$\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix} \begin{vmatrix} 5 & 0 \\ 2 & 3 \\ 1 & -1 \end{vmatrix} = 60 + 0 + 0 - (0 - 0 - 0) = 60$$

Or directly $5 * 3 * 4 = 60$

F-6: If each element of a row of a matrix is multiplied by a constant C and the results added to a different row, the determinant is not changed.

EX-14 Show that $|A|=|B|$ if $A = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{bmatrix}$ and B is the matrix

resultant from multiplying row (1) by 2 and adding to row (3).

i.e. $B = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{bmatrix}$

Sol.-

$|A| = 29$ from ex-10

$$|B| = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 0 \\ 5 & 0 \end{vmatrix} = 0 + 15 + 0 - (0 - 0 - 14) = 29$$

$\therefore |A| = |B|$

EX-15 Find
$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

Sol.-

$$\begin{array}{l} \xrightarrow{-2R_1+R_2} \\ \Rightarrow \Rightarrow \Rightarrow \end{array} \left| \begin{array}{cccc} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{array} \right| \quad \text{Expanding the determinant by using the}$$

$$\text{first column. } \Rightarrow \Rightarrow \Rightarrow 1 \cdot \begin{vmatrix} 5 & 2 & 0 & 5 & 2 \\ 0 & 4 & -1 & 0 & 4 \\ 1 & -6 & 1 & 1 & -6 \end{vmatrix} = 20 + 6 + 0 - (0 - 10 + 0) = 36$$

Linear Equations

There are many methods to solve a system of linear equations:

$$AX=B$$

I) Row Reduction method It is often possible to transform the linear equations step by step into an equivalent system of equations that is so simple it can be solved by inspection.

We start with $n \times (n+1)$ matrix $[A:B]$ whose first n columns are the columns of A and whose last column is B . We are going to transform this augmented matrix with a sequence of elementary row operations into $[I:S]$ where S is the solution of X .

$$2x + 3y - 4z = -3$$

EX-16 Solve the following linear equation: $x + 2y + 3z = 3$

$$3x - y - z = 6$$

Sol.

$$AX = B \quad \text{where} \quad A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 3 & -4 & -3 \\ 1 & 2 & 3 & 3 \\ 3 & -1 & -1 & 6 \end{array} \right] \xrightarrow[\begin{array}{l} -2R_2+R_1 \\ -3R_2+R_3 \end{array}]{\Rightarrow \Rightarrow \Rightarrow} \left[\begin{array}{ccc|c} 0 & -1 & -10 & -9 \\ 1 & 2 & 3 & 3 \\ 0 & -7 & -10 & -3 \end{array} \right]$$

$$\begin{array}{l} \text{inter change} \\ \Rightarrow \Rightarrow \Rightarrow \\ R_1 \text{ and } R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & -7 & -10 & -3 \end{array} \right] \xrightarrow[\begin{array}{l} 2R_2+R_1 \\ -7R_2+R_3 \end{array}]{\Rightarrow \Rightarrow \Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -17 & -15 \\ 0 & -1 & -10 & -9 \\ 0 & 0 & 60 & 60 \end{array} \right]$$

$$\begin{array}{l} \frac{17}{60}R_3+R_1 \\ \Rightarrow \Rightarrow \Rightarrow \\ \frac{1}{6}R_3+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 60 & 60 \end{array} \right] \xrightarrow[\begin{array}{l} R_2*(-1) \\ R_3*\frac{1}{60} \end{array}]{\Rightarrow \Rightarrow \Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right] = [I:S]$$

$$\text{Hence } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x=2, y=-1, z=1}$$

II) Cramer's Rule When the determinant of the coefficient matrix A of the system $AX=B$ is not zero (i.e. $|A| \neq 0$) the system has a unique solution that it may be found from the formulas:

$$X_i = \frac{|A_i|}{|A|} \quad \text{Where } |A_i| \text{ is the determinant of the matrix, comes from replacing the } i \text{th column in } A \text{ by the column of constant } B.$$

EX-17 Resolve example 16 using Cramer's rule:

Sol.

$$|A| = \begin{vmatrix} 2 & 3 & -4 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & -1 & -1 & 3 & -1 \end{vmatrix} = -4 + 27 + 4 - (-24 - 6 - 3) = 60$$

$$|A_1| = \begin{vmatrix} -3 & 3 & -4 & -3 & 3 \\ 3 & 2 & 3 & 3 & 2 \\ 6 & -1 & -1 & 6 & -1 \end{vmatrix} = 6 + 54 + 12 - (-48 + 9 - 9) = 120$$

$$|A_2| = \begin{vmatrix} 2 & -3 & -4 & 2 & -3 \\ 1 & 3 & 3 & 1 & 3 \\ 3 & 6 & -1 & 3 & 6 \end{vmatrix} = -6 - 27 - 24 - (-36 + 36 + 3) = -60$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -3 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & -1 & 6 & 3 & -1 \end{vmatrix} = 24 + 27 + 3 - (-18 - 6 + 18) = 60$$

$$\begin{aligned} \therefore x &= \frac{|A_1|}{|A|} = \frac{120}{60} \Rightarrow \boxed{x=2} \\ y &= \frac{|A_2|}{|A|} = \frac{-60}{60} \Rightarrow \boxed{y=-1} \\ z &= \frac{|A_3|}{|A|} = \frac{60}{60} \Rightarrow \boxed{z=1} \end{aligned}$$

The same result in ex - 16.

Problems – 8

1) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$. Find:-

- a) AB b) DC c) $(D+I)C$ d) $DC+C$ e) DCB
 f) EI g) $3A+E$ h) $-5E+A$ i) $E(2B)$

$\left(\begin{array}{l} \text{ans.: a) } \begin{bmatrix} -1 & 10 & -1 \\ -4 & 16 & -8 \end{bmatrix} \text{ b) } \begin{bmatrix} 3 & 9 \\ 4 & 3 \\ -4 & 3 \end{bmatrix} \text{ c, d) } \begin{bmatrix} 6 & 10 \\ 8 & 2 \\ -4 & 5 \end{bmatrix} \text{ e) } \begin{bmatrix} -6 & 42 & -9 \\ 1 & 20 & 6 \\ -7 & 4 & -18 \end{bmatrix} \\ \text{f) } \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} \text{ g) } \begin{bmatrix} 6 & 5 \\ 4 & 14 \end{bmatrix} \text{ h) } \begin{bmatrix} -14 & 7 \\ -20 & -6 \end{bmatrix} \text{ i) } \begin{bmatrix} 8 & 4 & 22 \\ 4 & 32 & 16 \end{bmatrix} \end{array} \right)$

2) Find the value of x :-

$$\begin{bmatrix} x & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ -7 \\ 5/4 \end{bmatrix} = 0$$

$\left(\text{ans.: } x = \frac{1}{2} \text{ or } x = 1 \right)$

3) Find v and w if: $\begin{bmatrix} 5 & w \end{bmatrix} = v \begin{bmatrix} -2 & 1 \end{bmatrix}$.

$\left(\text{ans.: } w = v = -\frac{5}{2} \right)$

4) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}$, Find:-

a) $2A + B'$

b) $B'A' - I$

$\left(\text{ans.: a) } \begin{bmatrix} 2 & -3 & 9 \\ 2 & 5 & 6 \end{bmatrix} \text{ b) } \begin{bmatrix} 10 & 19 \\ -5 & -6 \end{bmatrix} \right)$

5) Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix}$, Find:-

$(2A - I)B'$ and show that $(AB)' = B'A'$

$$\left(\text{ans.: } \begin{bmatrix} 5 & -1 \\ -2 & 11 \\ -6 & 26 \end{bmatrix} \right)$$

6) For what value of x will: $\begin{vmatrix} x & x & 1 \\ 2 & 0 & 5 \\ 6 & 7 & 1 \end{vmatrix} = 0$?
(ans.: $x = 2$)

7) Let A be an arbitrary 3 by 3 matrix and let R_{12} be the matrix obtained from the 3 by 3 identity matrix by interchanging row 1 and 2 : $R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. a) Compute $R_{12}A$

and show that you would get the same result by interchanging rows 1 and 2 of A . b) Compute AR_{12} and show that the result is that you would get by interchanging column 1 and 2 of A .

$$\left(\text{ans.: a) } \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ b) } \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \right)$$

8) Solve the following determinants:-

$$a) \begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad b) \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix} \quad c) \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$d) \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} \quad e) \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \quad f) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$g) \begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & - & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix} \quad h) \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$\left(\text{ans.: a) } -5 \quad b) 0 \quad c) -7 \quad d) 6 \right) \\ \left(\quad e) 38 \quad f) 1 \quad g) 2 \quad h) -1 \right)$$

9) Solve the following system of equations:-

$$\begin{array}{lll}
 a) \ x + 8y = 4 & b) \ 2x + 3y = 5 & c) \ x + y + z = 2 \\
 3x - y = -13 & 3x - y = 2 & 2x - y + z = 0 \\
 & & x + 2y - z = 4
 \end{array}$$

$$\begin{array}{lll}
 d) \ 2x + y - z = 2 & e) \ 2x - 4y = 6 & f) \ x - z = 3 \\
 x - y + z = 7 & x + y + z = 1 & 2y - 2z = 2 \\
 2x + 2y + z = 4 & 5y + 7z = 10 & 2x + z = 3
 \end{array}$$

$$\begin{array}{ll}
 g) \ x_1 + x_2 - x_3 + x_4 = 2 & h) \ 2x - 3y + 4z = -19 \\
 x_1 - x_2 + x_3 + x_4 = -1 & 6x + 4y - 2z = 8 \\
 x_1 + x_2 + x_3 - x_4 = 2 & x + 5y + 4z = 23 \\
 x_1 + x_3 + x_4 = -1 &
 \end{array}$$

$$\left(\begin{array}{ll}
 \text{ans. : } a) \ x = -4, y = 1 & b) \ x = y = 1 \\
 c) \ x = \frac{6}{7}, y = \frac{10}{7}, z = -\frac{2}{7} & d) \ x = 3, y = -2, z = 2 \\
 e) \ x = 0, y = -\frac{3}{2}, z = \frac{5}{2} & f) \ x = 2, y = 0, z = -1 \\
 g) \ x_1 = 2, x_2 = 0, x_3 = x_4 = -\frac{3}{2} & h) \ x = -2, y = 5, z = 0
 \end{array} \right)$$