Ordinary Differential Equations

- First order Differential Equations (FODE):

is defined by an equation $\frac{dy}{dx} = f(x,y)$ of two variables x and y.

Types of FODE:

1- Variable separation
2- Homogenous eq.
3- Exact eq.
4- linear eq.
5- Bernoulli's eq.

Applications of FODE.

- Newton's law of cooling
- Growth and decay
- Electrical circuts
- Falling body problems
- Dilution problems.

1. Variable Separation

This form is:

$$\int f(x)\,dx + \int g(y)\,dy = c$$

Ex.1/ solve; $(x^2+1)\dfrac{dy}{dx} = xy$.

Sol.

$$(x^2+1)\,dy = xy\,dx \qquad (\div (x^2+1)y)$$

$$\frac{(x^2+1)}{(x^2+1)y}\,dy = \frac{xy}{(x^2+1)y}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{x}{(x^2+1)}\,dx$$

$$\ln y = \frac{1}{2}\ln(x^2+1) + c$$

$$\ln|y| = \ln(x^2+1)^{\frac{1}{2}} + c \qquad (\text{take exp.})$$

$$y = \sqrt{x^2+1} * A \qquad (A = e^c)$$

$$\therefore y = A\sqrt{x^2+1}$$

Ex.2/ Find the solution of $e^{x}\frac{dy}{dx}=4$, that subjected to $y(0)=3$.

Solution:

$$e^{x}\,dy=4\,dx$$

$$dy=\frac{4}{e^{x}}\,dx$$

$$\int dy=\int 4e^{-x}\,dx$$

$$\therefore y=-4e^{-x}+c$$

B.C.

$$3=-4e^{0}+c \Longrightarrow c=7$$

$$\therefore y=-4e^{-x}+7$$

2. Homogenous eq.

It can be solved by introducting a new dependent variable.

$\rightarrow y = u x \quad \rightarrow \quad dy = u\,dx + x\,du$

EX.1/ solve; $2x\,dy = (x+y)\,dx$

solution:

$y = ux$ and $dy = u\,dx + x\,du$] sub. in D.E

$2x(u\,dx + x\,du) = (x + ux)\,dx$

$2x(u\,dx + x\,du) = x(1+u)\,dx \quad]\div x$

$2u\,dx + 2x\,du = dx + u\,dx$

$u\,dx + 2x\,du = dx$

$2x\,du = dx - u\,dx$

$2x\,du = (1-u)\,dx \quad]\div 2x(1-u)$

$$\int \frac{1}{(1-u)}\,du = \int \frac{1}{2x}\,dx \Rightarrow -\ln(1-u) = \frac{1}{2}\ln x + C$$

$$\ln(1-u)^{-1} = \ln x^{\frac{1}{2}} + \ln C$$

$$\ln\left(\frac{1}{1-u}\right) = \ln|\sqrt{x}\cdot c| \quad] \text{ take exp.}$$

$$\Rightarrow \frac{1}{1-u} = c\sqrt{x}$$

$y = ux$

$\rightarrow u = \frac{y}{x}$

$$\therefore \frac{1}{1-\frac{y}{x}} = c\sqrt{x}$$

Ex. 2/ Solve; $\frac{dy}{dx} = \frac{xy}{x^2-y^2}$

Solution: $xy\,dx = (x^2-y^2)\,dy$

$y = ux$ and $dy = x\,du + u\,dx$] sub in D.E.

$ux^2\,dx = (x^2 - u^2x^2)(x\,du + u\,dx)$

$ux^2\,dx = x^2(1-u^2)(x\,du + u\,dx) \quad (\div x^2)$

$\cancel{u\,dx} = x\,du + \cancel{u\,dx} - xu^2\,du - u^3\,dx$

$u^3\,dx = x\,du - xu^2\,du$

$u^3\,dx = x(1-u^2)\,du \quad (\div u^3x)$

$\rightarrow \frac{1}{x}\,dx = \frac{1-u^2}{u^3}\,du$

$\int \frac{1}{x}\,dx = \int \frac{1}{u^3}\,du - \int \frac{1}{u}\,du$

$\ln|x| = \frac{-1}{2u^2} - \ln|u| + c$

$\ln|x| = \frac{-1}{2\frac{y^2}{x^2}} - \ln\left|\frac{y}{x}\right| + c$

$\cancel{\ln|x|} = \frac{-x^2}{2y^2} - \ln|y| + \cancel{\ln|x|} + c$

$\ln|y| = \frac{-x^2}{2y^2} + c \Longrightarrow y = e^{\frac{-x^2}{2y^2}} * e^c \quad \{A = e^c$

$\therefore \quad y = A\,e^{\frac{-x^2}{2y^2}}$

3- Exact eq.

Standard form: $M(x,y)\,dx + N(x,y)\,dy = 0$ ---- ①

Eq. 1 is Exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Ex.1/ Solve $(2xy^2 - 4)\,dx + (2x^2y + 3)\,dy = 0$

Salution:

$\frac{\partial M}{\partial y} = 4xy$ and $\frac{\partial N}{\partial x} = 4xy \Rightarrow$ Exact eq.

$\frac{\partial f}{\partial x} = 2xy^2 - 4 \longrightarrow \int \partial f = \int (2xy^2 - 4)\,dx$

$f = x^2y^2 - 4x + g(y)$

$\frac{\partial f}{\partial y} = 2x^2y + g'(y)$

$\cancel{2x^2y} + 3 = \cancel{2x^2y} + g'(y)$

$g'(y) = 3 \longrightarrow g(y) = 3y$

$\therefore f(x,y) = x^2y^2 - 4x + 3y$

Ex 2/ Solve; $(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)\,dy = 0$

Sol:

$m = 4x^3y^3 - 2xy \Rightarrow \frac{\partial m}{\partial y} = 12x^3y^2 - 2x$

$N = 3x^4y^2 - x^2 \Rightarrow \frac{\partial N}{\partial x} = 12x^3y^2 - 2x$

$\therefore$ Exact equation

$\frac{\partial f}{\partial x} = 4x^3y^3 - 2xy \Rightarrow f = x^4y^3 - x^2y + g(y)$

$\frac{\partial f}{\partial y} = 3x^4y^2 - x^2 + g'(y)$

$\cancel{3x^4y^2} - \cancel{x^2} = \cancel{3x^4y^2} - \cancel{x^2} + g'(y)$

$\therefore g'(y) = 0$

$g(y) = 0$

$\therefore f(x,y) = x^4y^3 - x^2y$

4. Linear eq.

General form: $\frac{dy}{dx} + P(x)\,y = Q(x)$; to solve it:-

1- Determine Integration Factor $I.F = e^{\int P(x)\,dx}$

2- Integrate and Solve $y(x) = \frac{1}{I.F}\int I.F * Q(x)\,dx + \frac{c}{I.F}$

EX.1/ Solve ; $\frac{dy}{dx} + \frac{3y}{x} = x^2$ at $y(1) = \frac{1}{6}$

Sol:

$P(x) = \frac{3}{x}$, $Q(x) = x^2$

$I.f = e^{\int \frac{3}{x}dx} \rightarrow I.f = e^{3\int \frac{1}{x}dx} \rightarrow I.f = e^{3\ln|x|}$

$\therefore If = x^3$

$y(x) = \frac{1}{x^3}\int x^3 * x^2\,dx + \frac{c}{x^3}$

$= \frac{1}{\cancel{x^3}}\left[\frac{x^{\cancel{6}\,3}}{6}\right] + \frac{c}{x^3}$

$y(x) = \frac{x^3}{6} + \frac{c}{x^3}$

$\frac{1}{6} = \frac{1^3}{6} + \frac{c}{1^3} \Longrightarrow c = 0$

$\therefore y = \frac{x^3}{6}$

Ex.2/ Solve the D.E. $x\frac{dy}{dx} - y = x^2\cos x$ at $y(\frac{\pi}{2}) = \pi$

Solution:

$x\frac{dy}{dx} - y = x^2\cos x \xrightarrow{\div x} \frac{dy}{dx} - \frac{y}{x} = x\cos x$

$I.f = e^{\int \frac{-1}{x}dx} \rightarrow I.f = e^{-\int \frac{1}{x}dx} \rightarrow I.f = e^{-\ln(x)}$

$\therefore I.f = \frac{1}{x}$

$y = x\left[\int \frac{1}{x} \cdot x\cos x\, dx\right] + x\,c$

$y = x\sin x + xc$

B.C.

$\pi = \frac{\pi}{2}\left(\sin\frac{\pi}{2} + c\right) \quad (*2)$

$2\cancel{\pi} = \cancel{\pi}\left(\sin\frac{\pi}{2} + c\right)$

$2 = \sin\frac{\pi}{2} + c$

$2 = 1 + c \longrightarrow \boxed{c = 1}$

$\therefore y = x\sin x + x$

5- Bernoulli's equation

Standard form: $\frac{dy}{dx} + p(x)\, y = Q(x)\, y^n$

to solve it, we set $Z = y^{1-n}$

Ex. Solve $\frac{dy}{dx} + \frac{1}{x} y = x y^2$

Sol:

$Z = y^{1-2} \rightarrow Z = y^{-1} \rightarrow Z = \frac{1}{y}$ and $y = \frac{1}{Z}$

$dy = \frac{-1}{Z^2}\, dZ$ (Sub in D.E)

$\frac{-1}{Z^2}\frac{dZ}{dx} + \frac{1}{xZ} = x\frac{1}{Z^2}$ $(* -Z^2)$

$\frac{dZ}{dx} - \frac{Z}{x} = -x$ (linear form)

$I.f = e^{\int \frac{-1}{x} dx} \rightarrow e^{-\ln x} \rightarrow I.f = \frac{1}{x}$

$Z(x) = x \int \frac{1}{x} * (-x)\, dx + cx$

$Z(x) = -x^2 + cx$

$\frac{1}{y} = -x^2 + cx$

$\therefore y = \frac{1}{-x^2 + cx}$