CHAPTER Four
Sequences and Series

Sequences and infinite series

Sequences:

Def. An infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0, $n_0=1$

The number a(n) is the nth term of the sequence. Or the term with index n.

Ex(1) The sequence $\langle a_n \rangle$ whose nth terms is defined by:

1. $a_n = n-1$

$a_1 = 0,\ \ a_2 = 1,\ \ a_3 = 2, \ldots\ldots.$

The sequence $\langle a_n \rangle = \langle 1,2,3,\ldots., n-1,\ldots. \rangle$

2. $a_n = \dfrac{1}{n},\ \ a_1 = 1,\ \ a_2 = \dfrac{1}{2},\ \ \ a_3 = \dfrac{1}{3}, \ldots$

The sequence $\langle a_n \rangle = \left\langle 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots., \dfrac{1}{n}, \ldots \right\rangle$

3. $a_n = (-1)^{n+1}\left(\dfrac{1}{n}\right)$

$a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{3}, \ldots..$

The sequence $\langle a_n \rangle = \left\langle 1, -\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}, \ldots (-1)^{n+1}\left(\dfrac{1}{n}\right), \ldots \right\rangle$

4. $a_n = (-1)^{n+1}\left(1 - \dfrac{1}{n}\right)$

$a_1 = 0,\ a_2 = \dfrac{1}{2},\ a_3 = \dfrac{2}{3},\ a_4 = \dfrac{-3}{4}, \cdots$

the sequence $\langle a_n \rangle = \left\langle 0, -\dfrac{1}{2}, \dfrac{2}{3}, \ldots, (-1)^{n+1}\left(1 - \dfrac{1}{n}\right), \ldots \right\rangle$

5. $a_n = \dfrac{n-1}{n}$

$a_1 = 0, a_2 = \frac{1}{2}, a_3 = \frac{2}{3}, \ldots\ldots.$

the sequence $\langle a_n \rangle = \left\langle 0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots, \dfrac{n-1}{n}, \ldots. \right\rangle$

We refer to the sequence whose nth term is a with the notation $\{a_n\}$ (the sequence a sub n)

Excercise

Find the first five terms of the following sequences:

1- $\left\langle \frac{2n-1}{5n+2} \right\rangle$ 2- $\left\langle \frac{2^n}{n^2} \right\rangle$ 3- $\left\langle \frac{2^n}{5^n} \right\rangle$ 4- $\left\langle \frac{1-(-1)^n}{n^3} \right\rangle$

5- $\left\langle \frac{(-1)^{n+1}}{(2n-1)!} \right\rangle$

Convergence and divergence

The sequence $\{a_n\}$ converges to the number L, if to every positive number $\varepsilon > 0$, there corresponds an integer N such that for all n

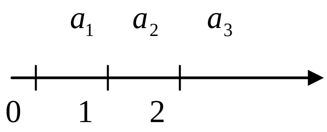
$n > N \Rightarrow |a_n - L| < \varepsilon$

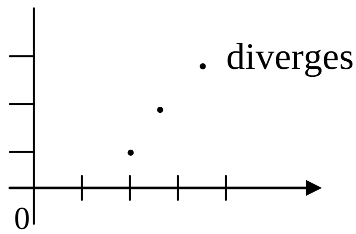
If no such limit exists, we say that $\{a_n\}$ diverges

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, and we call L the limit of the sequence.

i.e A sequence that has a limit is said to be converges and it is converges to that limit.

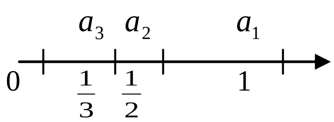
Ex (2) : $a_n = n - 1$

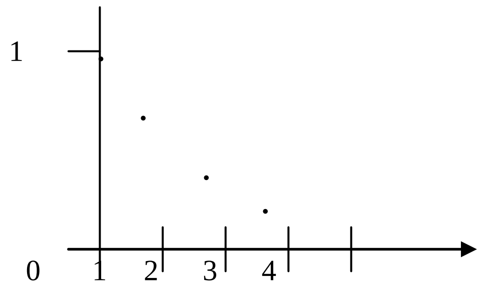




the sequence $\{a_n\}$ diverges

$a_n = \frac{1}{n}$





The sequence converges to 0.

The sequences are graphed here in two different ways by plotting the number a_n on the horizontal axis, and by plotting the points (n, a_n) in the coordinate plane.

Theorem: suppose that f(x) is a function defined for all $x \ge n_0$ and $\{a_n\}$ is a sequence such that $a_n = f(n)$ when $n \ge n_0$ if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty} a_n = L$

Ex(3) : Find the following limits :

1. $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1\lim_{n\to\infty}\frac{1}{n} = -1\cdot 0 = 0$

2. $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}1 - \lim_{n\to\infty}\frac{1}{n} = 1\cdot 0 = 0$

3. $\lim_{n\to\infty}\frac{5}{n^2} = 5\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 5\cdot 0\cdot 0 = 0$

4. $\lim_{n\to\infty}\frac{4-7n^3}{n^3+1} = \lim_{n\to\infty}\frac{\frac{4}{n^3}-7}{1+\frac{1}{n^3}} = \frac{0-7}{1+0} = -7$

Ex(4): Find the following limits by using L'Hopital rule:

1. $\lim_{n\to\infty}\frac{2^n}{5n}$

$$\lim_{n\to\infty}\frac{2^n}{5n} = \lim_{n\to\infty}\frac{2^n\cdot\ln 2}{5} = \infty$$

2.

Infinite Series

Def. If $\{a_n\} \equiv a_1, a_2, a_3, \ldots, a_n, \ldots$ is a given sequence and if s_n is defined by $s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$, then the sequence $\{s_n\} \equiv s_1, s_2, s_3, \ldots, s_n, \ldots$ is called an infinite series where

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

Notes:

1. The sequence $\{s_n\}$ is denoted by $\sum_{n=1}^{\infty} a_n$

2. $s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$

3. The number a_n is called the n^{th} term of the series $\sum_{n=1}^{\infty} a_n$

4. The series $\sum_{n=1}^{\infty} a_n$ is said to be converge to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k = \sum_{n=1}^{\infty} a_n$$

If no such limit exist then the series $\sum_{n=1}^{\infty} a_n$ is diverges.

Def (Geometric Series)

The series $a + ar + ar^2 + \ldots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ is called a geometric series where r is the ratio of any term to the one before it, and $a \neq 0$.

Theorem (1):

If $|r| < 1 (-1 < r < 1)$,the geometric series converges to the number $\frac{a}{1-r}$.

If $|r| \geq 1 (r \leq -1 or\ r \geq 1)$,the geometric series is diverges .

If r=0 , the series converges to 0.

That is $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \cdots = \frac{a}{1-r}$ if –1<r<1

Proof

case (1) if r= 1 , a≠ 0

$s_n = a + a + a + ... + a = na$

$\lim s_n = \lim na = a \lim n = \infty$

i.e if r =1, then $\langle s_n \rangle$ is diverges.

Case(2) if r≠1

$s_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$ ....(1)

$r s_n = ar + ar^2 + ar^3 + ... + ar^{n-1}$ ....(2)

$s_n - r s_n = a - ar^n$

(1-r) $s_n = a - ar^n$

$s_n =$ a(1-r^n)/ 1-r

If I r I ≤1, then $\langle r^n \rangle$ converges to 0.

Since $\sum_{n=1}^{\infty} ar^{n-1} = \lim s_n = \lim a/1-r$ (1- r^n)

=a/1-r [$\lim 1 - \lim r^n$]=a/1-r[1-0] =a/1-r

Thus $\sum_{n=1}^{\infty} ar^{n-1}$ converges to a/1-r ,when I rI<1

Case(3) if a≠ 0 , and Ir I>

$\therefore \lim a_n = \lim a/1-r(1-r^n) = a/1-r \lim(1-r^n)$ diverges.

Since $r^n \to \infty$ if a > 0

$r^n \to -\infty$ if a< 0

Case (4) if a≠0 , r = -1

$\lim a_n = \lim a/1-r(1-(-1)^n = a/1-r\lim[1-(-1)^n]$ diverges.

Case (5) if a=0

$\therefore \lim a_n = \lim \sum a r^{n-1} = \lim 0 = 0$

$\therefore \lim a r^{n-1}$ is diverges to 0 .

We get if -1< r< 1 then the series $\sum a r^{n-1}$ is converges to a/1-r

And if ≥r ro 1≤r)، 1≤ ׀r׀-1) the series $\sum a r^{n-1}$ is diverges.

Ex(8) Find the geometric series with a=1, r=3

$$1+3+9+27+\cdots+\cdots=1(1+3+9+\cdots)=\frac{1}{1-3}=\frac{1}{-2}$$

Ex(9) Find the geometric series with a=4, r=$-\frac{1}{2}$

$$4-2+1-\frac{1}{2}+\frac{1}{4}-\cdots=4\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots\right)=\frac{4}{1+\frac{1}{2}}=\frac{8}{3}$$

Ex(10):- Determine which of the following series is converges and which is diverges ?

1. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$ **2.** $\sum_{n=1}^{\infty}4\left(\frac{-1}{2}\right)^{n-1}$ **3.** $\sum_{n=1}^{\infty}\frac{2}{3^{n-1}}$

Solu.

1. The series $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$ is converge because it is a geometric series with a=1,r=1/2

$$=1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots+\left(\frac{1}{3}\right)^{n-1}+\cdots=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

2. The series $\sum_{n=1}^{\infty}4\left(\frac{-1}{2}\right)^{n-1}$ is converge because it is a geometric series with a=4, r=-1/2

$$=4-4\left(\frac{1}{2}\right)+4\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)^3+\cdots+4(-1)^{n-1}\left(\frac{1}{2}\right)^{n-1}+\cdots=\frac{4}{1-\left(-\frac{1}{2}\right)}=\frac{4}{1+\frac{1}{2}}=\frac{8}{3}$$

3. The series $\sum_{n=1}^{\infty}\frac{2}{3^{n-1}}$ is convergence because it is a geometric series with a=2, $r=\frac{1}{3}$

$$2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\left(\frac{2}{3^{n-1}}\right)=\frac{a}{1-r}=\frac{2}{1-\frac{1}{3}}=\frac{2}{\frac{2}{3}}=3$$

Ex(11) Find $\sum_{n=1}^{\infty}1/5^{n+1}$

$$\sum_{n=1}^{\infty} 1/5^{n+1} = 1/5^2 + 1/5^3 + 1/5^4 + \ldots\ldots.$$

$1/5^3 \times 5^2 = 1/5 = r$

$\therefore$ the series is geometric with a= $1/5^2$, r = 1/5

$\because$ -1< r < 1

$\therefore$ The series is converges to a/1-r

$$\therefore \sum_{n=1}^{\infty} 1/5^{n+1} = 1/5^2/1-1/5 = 1/5^2 \times 5/4 = 1/20$$

Ex(12) Find $\sum_{n=1}^{\infty} 7/4^n$

$$\sum_{n=1}^{\infty} 7/4^n = 7/4 + 7/4^2 + 7/4^3 + \ldots\ldots.$$

$7/4^2 \times 4/7 = 1/4$ =r

$\therefore$ the series is geometric with a= 7/4 , r = 1/4

$\because$ -1<r < 1

The series converges to a/1-r

$$\therefore \sum_{n=1}^{\infty} 7/4^n = a/1-r = 7/4/(1-1/4) = 7/4/(3/4) = 7/3$$

$\therefore$ The series is converges to 7/3

Ex(13) Find $\sum_{n=1}^{\infty} (-1)^n 5/4^n$

$\sum_{n=1}^{\infty} (-1)^n 5/4^n$ =5-5/4+5/ 4^2 - 5/ 4^3 +...........

-5/4 $\times$ 1/5=-1/4 =r

$\therefore$ the series is geometric with a= 5 , r = -1/4

$\because$ -1< r < 1

$\therefore$ the series is converges to a/1-r

$\therefore \sum_{n=1}^{\infty} (-1)^n 5/4^n$ = a/1-r= 5/1+4 =5/ 5/4=20/5 =4

The series converges to 4

Ex(14) Find $\sum_{n=1}^{\infty} 2^n/5^n$

$\sum_{n=1}^{\infty} 2^n/5^n$ = 2/5 + $2^2/5^2 + 2^3/5^3 + \ldots\ldots.$

$2^2/5^2 \times 5/2 = 2/5$ =r

$\therefore$ the series is geometric with a= 2/5 , r = 2/5

$\because$ -1 <r < 1

Then the series is converges to a/1-r

$\sum_{n=1}^{\infty} 2^n/5^n$ = a/1-r = 2/5 /1- (2/5) =(2/5)/(3/5) = 2/3

The series converges to 2/3

Taylor's And Maclaurian's Series Expansion

Suppose that f(x) and its derivatives $f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x), \ldots$ are all exist and continuous at x=a in some interval containing the point a, then f(x) can be written as

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)$$

$$(x-a)^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n \;------------------(1)$$

equation (1) is called Taylor's series expansion of f(x) at x=a

In the case when a=0, the equation(1) becomes

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots --- (2)$$

equation (2) is called Maclaurian's series of f(x) at x=0.

Ex(1):- Expand $f(x) = x^4 + 3x^3 + x^2 + 2x - 3$ about x=1

Solu. a=1

$$f(x) = f(1) + \frac{1}{1!}f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3 + \cdots + \frac{1}{n!}f^{(n)}(1)$$

$$(x-1)^n + \cdots$$

$$\begin{array}{lcl} f(x) = x^4 + 3x^3 + x^2 + 2x - 3 & & f(1) = 4 \\ f'(x) = 4x^3 + 9x^2 + 2x + 2 & & f'(1) = 17 \\ f''(x) = 12x^2 + 18x + 2 & \Rightarrow & f''(1) = 32 \\ f^{(3)}(x) = 24x + 18 & & f^{(3)}(1) = 42 \\ f^{(4)}(x) = 24 & & f^{(4)}(1) = 24 \\ f^{(5)}(x) = f^{(6)}(x) = \cdots = 0 & & \end{array}$$

$$f(x) = 4 + \frac{17}{1!}(x-1) + \frac{32}{2!}(x-1)^2 + \frac{42}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 4 + 17(x-1) + 16(x-1)^2 + 7(x-1)^3 + (x-1)^4$$

Ex(2):- Find Taylor series expansion of $f(x) = x^2 + 3x - 2$ about x=1, x=-1, x=2.

Solu.

$$\begin{array}{lclll} f(x) = x^2 + 3x - 2 & & f(1) = 2 & f(-1) = -4 & f(2) = 8 \\ f'(x) = 2x + 3 & \Rightarrow & f'(1) = 5 & f'(-1) = 1 & f'(2) = 7 \\ f''(x) = 2 & & f''(1) = 2 & f''(-1) = 2 & f''(2) = 2 \\ f^{(3)}(x) = 0 & & & & \end{array}$$

$$f(x) = 2 + 5(x-1) + (x-1)^2$$

$$f(x) = -4 + (x+1) + (x+1)^2$$

$$f(x) = 8 + 7(x-2) + (x-2)^2$$

Ex(3):- Find Taylor series expansion of $f(x) = \dfrac{1}{x}$ about x=1

Solu.

$$\begin{aligned}
f(x) &= x^{-1} & \qquad & f(1) = 1 \\
f'(x) &= -x^{-2} & & f'(1) = -1 \\
f''(x) &= 2x^{-3} & \Rightarrow & f''(1) = 2 \\
f'''(x) &= -6x^{-4} & & f'''(1) = -6 \\
f^{(4)}(x) &= 24x^{-5} & & f^{(4)}(1) = 24 \\
&\vdots & & \vdots
\end{aligned}$$

$$f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Ex(4):- Find Maclaurian series expansion of the functions $\sin x$

Solu.

$$\begin{aligned}
f(x) &= \sin x & \qquad & f(0) = 0 \\
f'(x) &= \cos x & & f'(0) = 1 \\
f''(x) &= -\sin x & \Rightarrow & f''(0) = 0 \\
f'''(x) &= -\cos x & & f'''(0) = -1 \\
f^{(4)}(x) &= \sin x & & f^{(4)}(0) = 0 \\
&\vdots & & \vdots
\end{aligned}$$

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Ex(5):- Find Maclaurian series expansion for the following functions

$$\frac{1}{1-x}, \frac{1}{(1-x)^2}$$

$$\begin{aligned}
f(x) &= \frac{1}{1-x} = (1-x)^{-1} & \Rightarrow \quad & f(0) = 1 \\
f'(x) &= (1-x)^{-2} = 1!(1-x)^{-2} & & f'(0) = 1! \\
f''(x) &= 2(1-x)^{-3} = 2!(1-x)^{-3} & & f''(0) = 2! \\
f'''(x) &= 6(1-x)^{-4} = 3!(1-x)^{-4} & & f'''(0) = 3! \\
f^{(4)}(x) &= 24(1-x)^{-5} = 4!(1-x)^{-5} & \Rightarrow \quad & f^{(4)}(0) = 4! \\
&\vdots & & \vdots \\
f^{(n)}(x) &= n!(1-x)^{-(n+1)} & & f^{(n)}(0) = n! \\
&\vdots & & \vdots
\end{aligned}$$

$$f(x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$f'(x)=(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+\cdots=\sum_{n=1}^{\infty}nx^{n-1}$$

$$f''(x)=2(1-x)^{-3}=2+6x+12x^2+20x^3+\cdots$$

$$\therefore (1-x)^{-3}=1+3x+6x^2+10x^3+\cdots$$

$$\frac{1}{1+x}=(1+x)^{-1}=[1-(-x)]^{-1}=1+(-x)+(-x)^2+(-x)^3+(-x)^4+\cdots$$

$$=1-x+x^2-x^3+x^4-\cdots$$

Ex(6):- Find Maclaurian series expansion for tanx .

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$$

Exercises:

1. Find the geometric series with

a. $a=4,\ r=\frac{1}{2}$ **b.** $a=\frac{1}{9},\ r=\frac{1}{3}$

2. Find the following series

a. $\sum_{n=1}^{\infty}\frac{1}{5^{n+1}}$ **b.** $\sum_{n=1}^{\infty}\frac{7}{4^n}$ **c.** $\sum_{n=1}^{\infty}\frac{3}{2^{n-1}}$

3. Determine whether the following series is converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{1}{n!}$ **b.** $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$

5. Find the Taylor series for the following functions at a=2

$$f(x)=\cos x$$

$$f(x)=\frac{1}{x}$$

6. Find Maclaurian series for the following functions

$$f(x)=\frac{1}{1-x} \qquad f(x)=\frac{1}{(1-x)^3}$$