عنوان المحاضرة:

Integration Methods: Integration by Parts, Integration by Partial Fractions

طرق التكامل: التكامل بالأجزاء، التكامل بالكسور الجزئية

Integration Methods :- طرق التكامل

① Integration By Parts :- التكامل بالاجزاء

This method is used when there is an association of two functions, one of which is not a derivative of the other.

وتستخدم طريقة التكامل بالاجزاء عندما يكون لدينا دالتين نريد تكاملها وليس لاحداهما مشتقة للاخرى مثل:-

a- $\int x e^x dx$

b- $\int t \sin t \, dt$

c- $\int e^\theta \cos\theta \, d\theta$

d- $\int x \ln x \, dx$

Derivative of Integration by parts Formula :- اشتقاق صيغة التكامل بالاجزاء

From the product rule of differentiation
من قاعدة مشتقة ضرب دالتين

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where u & v are both functions of x
حيث كل من u و v هم دالتين للمتغير x.

Rearranging gives;

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both side with respect to (w.r.t) x, yields,

$$\int u\frac{dv}{\cancel{dx}}\cancel{dx} = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{\cancel{dx}}\cancel{dx}$$

$$\int u\,dv = uv - \int v\,du$$ ← Integration by parts Formula

جزء اشتقاقه — جزء تكامله

* Now, which part to make equal to $\underline{u}$ & which to make equal to $\underline{v}$??

- السؤال هو، اي جزء من الدالتين يمكن اختيارها u والاخرى v ؟

To answer this enquiry, the choice must be such that the "u part" becomes a constant after successive differentiation, & the "dv part" can be integrated from standard integrals.

i.e

* x, t^2, or 3θ (algebric terms) $\xrightarrow[\text{to be}]{\text{chosen}}$ "u part"

Note!! (ملاحظة مهمة)

The only exception to this rule is when a "$\ln x$" term is involved; in this case $\ln x$ is chosen as the "u part"

● الاجابة على هذا الاستفسار، الاختيار يجب ان يكون بحيث يصبح الجزء u ثابتاً بعد اشتقاقه بشكل متكرر، و جزء dv يمكن تكامله بطرق التكامل القياسية المعروفة.

Examples

① Determine $\int x \cos x \, dx$

Solution

Let $u = x \longrightarrow du = dx$
let $dv = \cos x \, dx \longrightarrow v = \int \cos x \, dx = \sin x$

$$\int u \, dv = uv - \int v \, du$$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x - (-\cos x) + C$$

$$\therefore \int x \cos x \, dx = \boxed{x \sin x + \cos x + C} \quad \text{Ans}$$

② Determine $\int 3t e^{2t} \, dt$

Solution

Let $u = 3t \longrightarrow du = 3 \, dt$
Let $dv = e^{2t} dt \longrightarrow v = \int e^{2t} dt = \frac{1}{2} e^{2t}$

substitute into $\int u \, dv = uv - \int v \, du$

$$\therefore \int 3t e^{2t} \, dt = (3t)\left(\tfrac{1}{2} e^{2t}\right) - \int \left(\tfrac{1}{2} e^{2t}\right)(3 \, dt)$$

$$= \frac{3}{2} t e^{2t} - \frac{3}{2} \int e^{2t} \, dt$$

$$= \frac{3}{2} t e^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + C$$

$$\int 3t e^{2t} \, dt = \boxed{\frac{3}{2} e^{2t}\left(t - \frac{1}{2}\right) + C} \quad \text{Ans}$$

(3) Evaluate $\int_0^{\pi/2} 2\theta \sin\theta \, d\theta$

Solution

Let $u = 2\theta \longrightarrow du = 2\,d\theta$

Let $dv = \sin\theta\, d\theta \longrightarrow v = \int \sin\theta\, d\theta = -\cos\theta$

substitute into $\int u\,dv = uv - \int v\,du$, yields,

$$\int_0^{\pi/2} 2\theta \sin\theta\, d\theta = (2\theta)(-\cos\theta) - \int(-\cos\theta)(2\,d\theta)$$

$$= -2\theta\cos\theta + 2\int \cos\theta\, d\theta$$

$$= -2\theta\cos\theta + 2\sin\theta \Big]_0^{\pi/2}$$

$$= -\cancel{2}\left(\frac{\pi}{\cancel{2}}\right)\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right) - \left[0 + 2\sin 0\right]$$

$$= (0 + 2) - (0 + 0)$$

$$\int_0^{\pi/2} 2\theta\sin\theta\, d\theta = \boxed{2} \quad \text{Ans}$$

(4) Evaluate $\int_0^1 5x\, e^{4x}\, dx$

Solution

Let $u = 5x \longrightarrow du = 5\,dx$

Let $dv = e^{4x}\,dx \longrightarrow v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

substitute into $\int u\,dv = uv - \int v\,du$, gives,

$$\int_0^1 5x\, e^{4x}\, dx = (5x)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)(5\,dx)$$

$$= \frac{5}{4}x e^{4x} - \frac{5}{4}\int e^{4x}\,dx$$

$$= \frac{5}{4}x e^{4x} - \frac{5}{4}\left(\frac{e^{4x}}{4}\right) = \frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right)\Big]_0^1$$

$$= \frac{5}{4}e^{4(1)}\left(1 - \frac{1}{4}\right) - \left[\frac{5}{4}\cancel{e^{0}}\left(0 - \frac{1}{4}\right)\right]$$

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$$\int_0^1 5x\,e^{4x}\,dx = \frac{15}{16}e^4 - \left(-\frac{5}{16}\right)$$

$$= 51.186 + 0.313 = \boxed{51.5} \quad \underline{\text{Ans}}$$

(5) Evaluate $\int x^2 \sin x\,dx$

solution

Let $u = x^2 \rightarrow du = 2x\,dx$

Let $dv = \sin x\,dx \Longrightarrow v = \int \sin x\,dx = -\cos x$

substitute into $\int u\,dv = uv - \int v\,du$, gives,

$$\int x^2 \sin x\,dx = (x^2)(-\cos x) - \int(-\cos x)(2x\,dx)$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

The integral of $\int x\cos x\,dx$ is not a "standard integral" & it can only be determined by using the integration by parts formula again,

وهنا نلاحظ بأن حد الـ $\int v\,du$ نتج تكامل غير قياسي، ولهذا نحتاج أن نستخدم التكامل بالأجزاء مرة اخرى وهنا الحد فقط.

From Example (1) $\int x\cos x\,dx = x\sin x + \cos x + C$

$$\therefore \int x^2\sin x\,dx = -x^2\cos x + 2[x\sin x + \cos x] + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$= \boxed{(2-x^2)\cos x + 2x\sin x + C} \quad \underline{\text{Ans}}$$

⑥ Find $\int x \ln x \, dx$

Solution

Let $u = \ln x \longrightarrow du = \frac{1}{x}\, dx$

let $dv = x\, dx \longrightarrow v = \frac{x^2}{2}$

substitute into $\int u\, dv = uv - \int v\, du$, yields,

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \left(\frac{x^2}{2}\right) \frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2}\, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \boxed{\frac{x^2}{4}(2\ln x - 1) + c} \quad \text{Ans}$$

⑦ Find $\int \ln x \, dx$

Solution

Let $u = \ln x \longrightarrow du = \frac{1}{x}\, dx$

Let $dv = dx \longrightarrow v = x$

substitute into $\int u\, dv = uv - \int v\, du$, gives,

$$\int \ln x \, dx = x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\, dx$$

$$= x \ln x - \int dx = x \ln x - x + c$$

$$= \boxed{x(\ln x - 1) + c} \quad \text{Ans}$$

② Integration By Partial Fractions:-

التكامل بالكسور الجزئية :-

If we do algebric addition of two fractions as $\frac{1}{x-2}+\frac{3}{x+1}$ then we take a common denominator of these two fraction as follows,

$$\frac{1}{(x-2)}+\frac{3}{(x+1)}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}=\frac{4x-5}{x^2-x-2}$$

So, the reverse process of moving from $\frac{4x-5}{x^2-x-2}$ to $\frac{1}{x-2}+\frac{3}{x+1}$ is called resolving into "partial fractions".

Examples/

① $\int \frac{11-3x}{x^2+2x-3}$

Solution,

By using a partial fractions, we can make the rational equation have a simple form as follows,

$$\frac{11-3x}{x^2+2x-3}=\frac{11-3x}{(x-1)(x+3)}=\frac{A}{(x-1)}+\frac{B}{(x+3)}$$

$$\frac{11-3x}{(x-1)(x+3)}=\frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$\therefore\ 11-3x=A(x+3)+B(x-1)$$

To determine constants A & B, values of x are chosen to make the term in A or B equal to zero.

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- when $x = 1$, then

$$11 - 3(1) = A(1+3) + B(0)$$
$$8 = 4A$$
$$\therefore \boxed{A = 4}$$

- when $x = -3$, then

$$11 - 3(-3) = A(0) + B(-3-1)$$
$$20 = -4B$$
$$\therefore \boxed{B = -5}$$

$$\therefore \frac{11 - 3x}{x^2 + 2x - 3} = \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

Hence, $$\int \frac{11 - 3x}{x^2 + 2x - 3}\, dx = \int \left\{ \frac{2}{(x-1)} - \frac{5}{(x+3)} \right\} dx$$

$$= \boxed{2 \ln(x-1) - 5 \ln(x+3) + C} \quad \text{Ans}$$

هنا نستطيع جميع التعابير التي من الممكن أن تأتي في الدالة الكسرية مع صيغ الكسور الجزئية المرافقة لها

Expression	Form of Partial Fraction
① $\frac{Fx}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)} + \frac{B}{(x-b)} + \frac{C}{(x+c)}$
② $\frac{Fx}{(x+a)^3}$	$\frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}$
③ $\frac{F(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)} \quad \frac{C}{(x+d)}$

ملاحظة:

لحل اي سؤال كسري بطريقة الكسور الجزئية نعمل التالي:

① يجب تحويل المقام الى معامل كما عملنا في المثال السابق.

② درجة البسط يجب أن تكون اقل من درجة المقام بمقدار واحد.

② $\int \frac{x^2+1}{x^2-3x+2}\,dx$

Solution

→ Here the denominator is of same degree as the numerator. Thus dividing out, yields,

$$x^2-3x+2 \overline{\big)\, x^2+1} \quad \begin{array}{r} 1 \\ \hline x^2+1 \\ -x^2 \mp 3x \mp 2 \\ \hline 3x-1 \end{array} \Longrightarrow 1+\frac{3x-1}{x^2-3x+2}$$

$$\therefore \int \frac{x^2+1}{x^2-3x+2}\,dx = \int\left(1+\frac{3x-1}{x^2-3x+2}\right)dx$$

$$= \int dx + \int \frac{3x-1}{x^2-3x+2}\,dx$$

هنا نحتاج تجزئة الحد الثاني وتحويله بشكل بسيط بواسطة استخدام الكسور الجزئية وبعدها نرجع الى التكامل.

$$\frac{3x-1}{x^2-3x+2} = \frac{3x-1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\therefore 3x-1 = A(x-2)+B(x-1)$$

- Let $x=1 \longrightarrow 3(1)-1 = A(1-2)+B(1-1)$

$$2 = -A + B(0)$$

$$\therefore \boxed{A=-2}$$

- Let $x=2 \longrightarrow 3(2)-1 = A(2-2)+B(2-1)$

$$5 = A(0)+B$$

$$\boxed{B=5}$$

Hence, $\frac{x^2+1}{x^2-3x+2} = 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$

$$\therefore \int \frac{x^2+1}{x^2-3x+2}\,dx = \int \left\{ 1 - \frac{2}{x-1} + \frac{5}{x-2} \right\} dx$$

$$= x - 2\ln(x-1) + 5\ln(x-2) + C$$

$$= \boxed{x + \ln\left(\frac{(x-2)^5}{(x-1)^2}\right) + C} \quad \underline{\text{Ans}}$$

③ Evaluate $\displaystyle\int^{3} \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx$

(Solution)

In this problem, the numerator is of higher degree than the denominator. Thus dividing out, gives

$$\begin{array}{r|l} & \boxed{x-3} \\ x^2+x-2 & x^3-2x^2-4x-4 \\ & \mp x^3 \mp x^2 \pm 2x \\ \hline & -3x^2-2x-4 \\ & \pm 3x^2 \pm 3x \mp 6 \\ \hline & \boxed{x-10} \end{array}$$

Thus $\dfrac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3 + \dfrac{x-10}{x^2+x-2}$

هنا يحتاج تجزئة الحد الثاني فقط. نحوله الى شكل الكسور الجزئية وبعدها نرجع الجميع تحت التكامل.

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} = x-3+\frac{-x-10}{x^2+x-2}$$

$$= x-3+\boxed{\frac{x-10}{(x+2)(x-1)}}$$

Let $\dfrac{x-10}{(x+2)(x-1)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-1)}$

$$= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equating the numerators, yields, يساوي البسط للطرفين نحصل على,

$$x-10 = A(x-1)+B(x+2)$$

let $x=-2 \Longrightarrow -2-10 = A(-2-1)+B(\cancel{-2+2})$

$$-12 = -3A$$

$$\therefore \boxed{A=4}$$

let $x=1 \longrightarrow 1-10 = A(\cancel{1-1})+B(1+2)$

$$-9 = 3B$$

$$\therefore \boxed{B=-3}$$

$$\therefore \boxed{\frac{x-10}{(x+2)(x-1)} = \frac{4}{(x+2)} + \frac{-3}{(x-1)}}$$

Back substitute, gives,

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} = x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}$$

Hence $\displaystyle\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx = \int_2^3\left(x-3+\frac{4}{x+2}-\frac{3}{x-1}\right)dx$

$$= \frac{x^2}{2}-3x+4\ln(x+2)-3\ln(x-1)\Big]_2^3$$

$$= \left(\frac{9}{2} - 9 + 4\ln 5 - 3\ln 2\right) - \left(2 - 6 + 4\ln 4 - 3\ln 1\right)$$

$$\simeq \boxed{-1.687} \quad \underline{\text{Ans}}.$$

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